Math 272a, homework 2

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Problem 1. Define a singular n-cube in a space X to be a continuous map $\sigma : I^n \to X$. Using $\mathbf{Z}/2$ coefficients for simplicity, define the cubical chain group of X, $C_n^c(X)$, to be the $\mathbf{Z}/2$ vector space with generators the singular n-cubes. Define a boundary map ∂, to make these cubical chain groups into a chain complex. Verify that $\partial\partial = 0$. Temporarily, let us denote by $H_n^c(X; \mathbf{Z}/2)$ the resulting 'homology groups' of X.

Show that we do not obtain the ordinary singular homology groups of X (with $\mathbf{Z}/2$ coefficients) by this recipe.

Compute $H_1^c(S^1; \mathbf{Z}/2)$.

(See the book on singular homology by Massey. He does singular homology using cubes, but modifies the above definition so as to recover the expected singular homology groups.)

Problem 2. Let X be a topogical space. The nth (unoriented) bordism group $MO_n(X)$ is defined as follows. An element of $MO_n(X)$ is represented by a pair (M, s), where M is a smooth, compact n-manifold (without boundary) and $s : M \to X$ is a continuous map. Two pairs (M, s) and (M', s') represent the same class if there is a cobordism, i.e. a smooth compact $(n+1)$-manifold N with boundary $\partial N = M \coprod M'$, and a continuous map $t : N \to X$ such that $t|M = s$ and $t|M' = s'$.

The group operation is disjoint union, and the zero element is the empty map. Verify the group axioms. Show that homotopic maps $f, g : X \to Y$ give rise to the same homomorphism $f_* = g_* : MO_n(X) \to MO_n(Y)$ (which you should define).

Let T be the torus $\mathbf{R}^2/\mathbf{Z}^2$. Let $s : T \to T$ be the map arising from the linear map on $\mathbf{R}^2$ given by the matrix

$$\begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}.$$

Show that (T, s) represents the zero element of $MO_2(T)$. Show that for any compact, smooth n-manifold M, the pair (M, id) represents a non-zero element of $MO_n(M)$.

Remark. You may assume that any continuous map $f : N \to M$ between smooth manifolds is homotopic to a smooth map $\tilde{f}$; and that if N has boundary and f is already smooth on ∂N, then we can arrange that $\tilde{f}|\partial N = f|\partial N$.

Peter Kronheimer