Math 272a, homework 3

October 4, 2003

Problem 1. *The Mayer-Vietoris sequence.* Let $K = (V, S)$ be an abstract simplicial complex (so S is a collection of non-empty finite subsets of V, closed under taking non-empty susbets). Suppose K is the union of two subcomplexes, $K^a = (V^a, S^a)$ $(a = 1, 2)$, which means that $V = V^1 \cup V^2$ and $S = S^1 \cup S^2$. Show that $K^1 \cap K^2$ is also an abstract simplicial complex. Show that there is a long exact sequence of simplical homology groups,

$$\cdots \to H_n(K^1 \cap K^2) \xrightarrow{x} H_n(K^1) \oplus H_n(K^2) \xrightarrow{y} H_n(K) \xrightarrow{\partial} H_{n-1}(K^1 \cap K^2) \to \cdots,$$

where $x(\alpha) = (i^1_*\alpha, i^2_*\alpha)$, $y(\beta, \gamma) = j^1_*\beta - j^2_*\gamma$, and the maps i^a and j^a are the obvious inclusions.

The real thing. If a topological space is written as a union, $X = X^1 \cup X^2$, and the interiors of X^1 and X^2 cover X, then we saw that the simplicial homology of X can be computed from the subcomplex $C^{\mathcal{U}}_\bullet(X) \subset C_\bullet(X)$ corresponding to the cover $\mathcal{U} = \{X^1, X^2\}$ of X, i.e the subcomplex generated by the singular simplices that are contained in at least one of X^1 and X^2. Using this fact, adapt the argument you used above to establish the Mayer-Vietoris sequence in singular homology,

$$\cdots \to H_n(X^1 \cap X^2) \xrightarrow{x} H_n(X^1) \oplus H_n(X^2) \xrightarrow{y} H_n(X) \xrightarrow{\partial} H_{n-1}(X^1 \cap X^2) \to \cdots.$$

Explain why the same sequence holds in reduced homology also.

Problem 2. Recompute $\tilde{H}_n(S^n)$ inductively using the Mayer-Vietoris sequence for reduced homology.

The *suspension* SX of a space X is the quotient space of $I \times X$ by the equivalence relation in which the two non-trivial equivalence classes are $\{1\} \times X$ and $\{0\} \times X$. Use the Mayer-Vietoris sequence to show that $\tilde{H}_n(SX) \cong \tilde{H}_{n-1}(X)$ for all n.

Problem 3. Use the Mayer-Vietoris sequence to compute the homology of a Klein bottle, by viewing the bottle as made from two Möbius bands joined along their boundary.

Problem 4. Let (C^n_*, ∂) be a collection of chain complexes indexed by $n \in \mathbf{Z}$. Let $f^n_* : C^n_* \to C^{n+1}_*$ be a chain map, one for each n. Suppose that the composite $f_{n+1} \circ f_n : C^n \to C^{n+2}$ is chain-homotopic to zero for all n, by a chain-homotopy $K^n : C^n_* \to C^{n+2}_{*+1}$; that is,

$$f^{n+1} \circ f^n = \partial K^n + K^n \partial.$$

Show that the map

$$\psi^n \stackrel{\text{def}}{=} f^{n+2} \circ K^n - K^{n+1} \circ f_n$$

is an anti-chain map from $C^n_* \to C^{n+3}_*$, meaning that $\partial \circ \psi^n = -\psi^n \circ \partial$, and deduce that ψ^n gives rise to map on homology,

$$\psi^n_* : H_i(C^n_*, \partial) \to H_{i+1}(C^{n+3}_*, \partial) \tag{1}$$

for all n and i. Finally, suppose that (1) is an isomorphism for all n and i. Deduce that the sequence

$$\cdots \longrightarrow H_i(C^n_*, \partial) \xrightarrow{f^n_*} H_i(C^{n+1}_*, \partial) \xrightarrow{f^{n+1}_*} H_i(C^{n+2}_*, \partial) \longrightarrow \cdots$$

is exact.

Problem 5. Suppose $K = (V, S)$ and $K' = (V', S')$ are abstract simplicial complexes, and suppose both vertex sets are ordered, so that we may define the simplicial chain complexes $C_*(K)$ and $C_*(K')$. Recall that $C_*(K)$ is a free abelian group with generators indexed by elements $s \in S$. To avoid confusion below, let us write e_s for the generator of $C_*(K)$ corresponding to s.

A *simplicial map* from K to K' is a map $f : V \to V'$ with the property that

$$s \in S \implies f(s) \in S'.$$

(Note however that $f(s)$ may have fewer elements than s.) Given such an f, define a map $f_\# : C_*(K) \to C_*(K')$ by

$$f_\#(e_s) = \epsilon \, e_{f(s)},$$

where $\epsilon = 0$ if $f(s)$ has fewer elements that s, and $\epsilon = \pm 1$ otherwise: we take the sign $+1$ if the bijection $f|_s : s \to f(s) \subset V'$ is even and -1 if it is odd. (We can make sense of "even" and "odd" because the elements of $s \subset V$ and $f(s) \subset V'$ are ordered.)

Show that $f_\#$ is a chain map.

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