Math 272a, homework 4

October 12, 2003

Problem 1. Let $C \subset S^n$ be a subset homeomorphic to (the topological realization of) a finite simplicial complex K of dimension q. Show that $\tilde{H}_p(S^n \setminus C) = 0$ for $p < n - q - 1$. Suppose further that the simplicial complex K has only one q-simplex. Show that $\tilde{H}_p(S^n \setminus C) = 0$ also for $p = n - q - 1$.

Remark. We will be able to deal much better with this later, when we have "Alexander duality" available. For now, I am looking for an answer which follows the material we developed in class this last week.

Problem 2. Let X be a CW complex with one 0-cell and infinitely many 1-cells. Show that the topology of X is not metrizable (we cannot make X into a metric space in such a way that its metric topology is its CW topology).

More generally, show that the same applies to any CW complex X in which some point belongs to the closure of infinitely many cells.

Problem 3.

(a) Let E be the set of unit tangent vectors to the unit sphere $S^2 \subset \mathbf{R}^3$. We can identify E with the set of orthonormal pairs of vectors $(\mathbf{u}, \mathbf{v})$ in 3-space. Let $F \subset E$ be the copy of the circle S^1 consisting of those pairs with $\mathbf{v} = (0, 0, 1)$, the set of unit tangent vectors at the north pole. Show that a map $F \to S^1$ of non-zero degree cannot extend to a map $E \to S^1$.

(b) Let $\sigma : S^1 \to S^1$ be rotation by $2\pi/3$. Show that a map $f : S^1 \to S^1$ having the property that $f \circ \sigma = \sigma \circ f$ must have non-zero degree.

(c) Combine the above two facts to prove the following result. In this statement, the surface of the earth is taken to be a perfect sphere, and the surface temperature (at a given time) is taken to be a continuous real-valued function. With these assumptions prove that (at any given time) there are three points on the earth's surface, each one 2,000 miles from the other two, where the surface temperatures are the same.

Remark. A stronger result is true: rather than consider equilateral triangles of side-length 2,000 miles, one can consider triangles congruent to any given triangle on the earth's surface, and the result will still hold. (This result is due to Floyd.) One can try and guess generalizations of the triangle result, increasing the dimension of the sphere, increasing the number of points, and replacing a single real-valued function (temperature) by an $\mathbf{R}^m$-valued function. For example, given a continuous real-valued function f on the n-sphere, and $n + 1$ points $s_1, \ldots, s_{n+1}$, is there an element ρ of the rotation group $SO(4)$ such that $f(\rho(s_1)), \ldots, f(\rho(s_{n+1}))$ are equal? The answer is not known except for the case of the circle and the 2-sphere (Floyd's result).

Knaster's conjecture asserts that the same thing holds for $\mathbf{R}^m$-valued functions on S^n and k points, where $k = n - m + 2$. Some cases are known to be true, for example the case $k = 2$ (the case $k = 1$ is trivial). But other cases are known to be false. For exmple, the conjecture is false for all cases with $k > 2$ and $m \geq 3$: look at the article "Counterexamples to Knaster's conjecture" by William Chen, in volume 37 of *Topology*. So the interesting remaining cases are:

(a) a single real function on S^n and $n + 1$ points (the case described above, where an affirmative answer is known only for $n = 2$);

(b) an $\mathbf{R}^2$ valued function on S^n, and n points (where again an affirmative answer is known only for $n = 2$).

Peter Kronheimer