Math 272a, homework 5

October 18, 2003

Problem 1. Using the standard CW structure, calculate the homology of $\mathbf{RP}^n$ with coefficients $\mathbf{Z}/6$.

Problem 2. Explain how to exhibit a homeomorphism between $SO(3)$ and $\mathbf{RP}^3$. Using this identification, describe the family of rotations in $SO(3)$ parametrized by a standard copy of $\mathbf{RP}^2$ in $\mathbf{RP}^3$. Consider the action of $SO(3)$ on S^2, and let $u : SO(3) \to S^2$ be the map $g \mapsto ge$, where $e \in S^2$ is a chosen basepoint. Calculate the resulting map

$$u_* : H_2(SO(3); G) \to H_2(S^2; G)$$

with coefficient group $G = \mathbf{Z}/2$ and with coefficient group $G = \mathbf{Z}/6$.

Problem 3. The CW complex X is obtained from the $2n$-dimensional CW complex $\mathbf{RP}^{2n}$ by attaching a single $2n+1$-cell. Show that the homology groups of X (with any coefficient group) are the same as those of $\mathbf{RP}^{2n+1}$.

Problem 4. In class, we described a CW structure for the n-torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$ in which there are $\binom{n}{k}$ k-cells for each k.

Calculate the cellular homology by showing that the boundary maps in the cellular chain complex are zero.

Let A be a linear transformation of $\mathbf{R}^n$ that maps $\mathbf{Z}^n$ to $\mathbf{Z}^n$. Such an A gives rise to map $a : T^n \to T^n$. Show that we can identify $H_1(T^n)$ with $\mathbf{Z}^n$ in such a way that the map $a_* : H_1(T^n) \to H_1(T^n)$ coincides with $A : \mathbf{Z}^n \to \mathbf{Z}^n$.

Problem 5. Let Y be a topological space, and $f : Y \to Y$ a homeomorphism. The *mapping torus* of f is the quotient space of $I \times Y$ by the relation that equates each $(0, y)$ with $(1, f(y))$. (So if f is the identity, the mapping torus is $S^1 \times Y$.) Show how to use Mayer-Vietoris to compute the homology of the mapping torus, in terms of $H_\bullet(Y)$ and f_*.

Problem 6. For each $n \geq 3$, exhibit an orientation-preserving self-diffeomorphism $f : T^{n-1} \to T^{n-1}$ of the $(n-1)$-torus, such that the mapping torus of f is an n-manifold with $H_1 = \mathbf{Z}$.

Peter Kronheimer