Math 272a, homework 6

October 25, 2003

Problem 1. Show that $H^1(X)$ is always torsion-free, for any X.

Problem 2. Prove the "ham sandwich theorem". That is, let f_1, f_2 and f_3 be three bounded, integrable functions on $\mathbf{R}^3$, each of which vanishes outside some compact set. Show that there is an affine plane Π cutting $\mathbf{R}^3$ into two half-spaces U^+ and U^-, such that

$$\int_{U^+} f_i(x, y, z)\; dx\, dy\, dz = \int_{U^-} f_i(x, y, z)\;\; dx\, dy\, dz$$

for $i = 1, 2, 3$. Explain why this is called the "ham sandwich theorem". (There's no need to try and explain why "ham" rather than, say, "cheese" however.)

Problem 3. Calculate $\mathrm{Ext}(\mathbf{Z}/p, \mathbf{Z}/q)$ for primes p and q, using a presentation of $\mathbf{Z}/p$. Calculate $H^n(\mathbf{RP}^7, \mathbf{Z}/6)$, (a) starting from your knowledge of $H_n(\mathbf{RP}^7)$ and using the result above, and then (b) more directly from the cellular cochain complex.

Suppose X is a space whose homology groups $H_k(X)$ are *finitely generated*. Calculate $\mathrm{Ext}(\mathbf{Z}/r, \mathbf{Z})$ to show that

$$H^n(X) \cong F_n(X) \oplus T_{n-1}(X),$$

where $T_k \subset H_k$ is the torsion subgroup and $F_k = H_k/T_k$ is the free quotient. (This is is not a natural isomorphism.)

Problem 4. Let $H = \mathbf{Z}[1/2]$ be the subgroup of $\mathbf{Q}$ consisting of rational numbers whose denominator is a power of 2. Give a presentation of H as the quotient of a free abelian group, and use this to get a description of $\mathrm{Ext}(H, \mathbf{Z})$. Show that $\mathrm{Ext}(H, \mathbf{Z})$ is isomorphic to the group $\mathbf{Z}_2$ of 2-adic integers.

Let M be the mapping cylinder of the degree-2 map $f : S^1 \to S^1$, $\theta \mapsto 2\theta$. (Look up mapping cylinders in Hatcher.) Describe a CW complex X with $H^2(X) = \mathbf{Z}_2$, by telescoping a countably infinite collection of copies of the space M.

Aside. The group $\mathbf{Z}_2$ of 2-adic integers has as elements the infinite sequences of 0s and 1s extending leftward:

$$\cdots 01000111010110010 11101.$$

The law of addition is "binary addition with carry". So the uncountable group $\mathbf{Z}_2$ has a countable subgroup isomorphic to $\mathbf{Z}$, consisting of sequences that are eventually 0 as you head leftward. We can describe $\mathbf{Z}_2$ as the completion of the metric space $\mathbf{Z}$, when $\mathbf{Z}$ is equipped with a metric in which the distance $d(n, m)$ for $n \neq m$ is $1/i$, where $i \geq 0$ and 2^i is the highest power of 2 that divides $n - m$.

Problems from Hatcher

Please do questions 5, 6 and 11 from section 3.1 of Hatcher's book.

Peter Kronheimer