

## Math 272a, homework 7

November 1, 2003

**Problem 1.** Here is another way to calculate the cohomology ring of  $\mathbf{CP}^n$ , starting from the Kunneth theorem. It is not a very good plan to do it this way, as the proof does not appear to adapt to the case of  $\mathbf{RP}^n$  or  $\mathbf{HP}^n$ .

Let  $V_k$  denote the vector space of homogeneous complex polynomials of degree  $k$  in 2 variables. It has as basis the monomials  $z_1^k, z_1^{k-1}z_2, \dots, z_2^k$ , and its dimension is  $k + 1$ . Consider the map

$$p : V_1 \times \cdots \times V_1 \rightarrow V_n$$

given by

$$p(f_1, \dots, f_n) = \prod f_i.$$

Show that it gives rise to a well-defined map

$$\pi : \mathbf{CP}^1 \times \cdots \times \mathbf{CP}^1 \rightarrow \mathbf{CP}^n.$$

Calculate the map  $\pi^* : H^i(\mathbf{CP}^n; \mathbf{Z}) \rightarrow H^i(\mathbf{CP}^1 \times \cdots \times \mathbf{CP}^1; \mathbf{Z})$ , for  $i = 2$  and for  $i = 2n$ . For the latter, you might want to think back to our discussion of the degree of a map between spheres. Use the Kunneth theorem (applied to the product of  $\mathbf{CP}^1$ 's) to deduce that the  $n$ th power of the generator of  $H^2(\mathbf{CP}^n; \mathbf{Z})$  is a generator of  $H^{2n}(\mathbf{CP}^n; \mathbf{Z})$ .

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**Problem 2.** If  $\alpha \in H^i(X; \mathbf{Z})$  and  $i$  is odd, show that  $2\alpha \smile \alpha = 0$ .

Let  $X$  be (the topological realization of) a simplicial complex. Show that every element  $\alpha$  in  $H^1(X; \mathbf{Z})$  has the form  $f^*(\alpha_1)$ , where  $\alpha_1 \in H^1(S^1; \mathbf{Z})$  is a generator and  $f : X \rightarrow S^1$  is a map. Deduce that  $\alpha \smile \alpha = 0$ .

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**Problem 3.** Suppose  $m < n$ , and let  $F : \mathbf{CP}^n \rightarrow \mathbf{CP}^m$  be any map. Show that the resulting map  $F_* : H_2(\mathbf{CP}^n) \rightarrow H_2(\mathbf{CP}^m)$  is zero.

Let  $d$  be a positive integer, and let  $f_1, \dots, f_n$  be homogeneous complex polynomials of degree  $d$  in  $n + 1$  variables. Use the first part to show that the simultaneous equations

$$f_i(z_0, z_1, \dots, z_n) = 0, \quad (i = 1, \dots, n)$$

have a non-zero solution  $(z_0, \dots, z_n)$  in  $\mathbf{C}^{n+1}$ . *Hint.* To calculate the map on  $H_2$ , you might look at the restriction of your  $F$  to a  $\mathbf{CP}^1$  inside  $\mathbf{CP}^n$  and deform  $F|_{\mathbf{CP}^1}$  by a homotopy until you have a map from  $\mathbf{CP}^1$  to  $\mathbf{CP}^1$ .

Establish the same result with the more general hypothesis, that  $f_i$  has degree  $d_i > 0$  (and the  $d_i$  are no longer all equal). This result is a weak version of *Bézout's theorem*.

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**Problem 4.** If  $f_i(x_0, \dots, x_n)$  is a *real* homogeneous polynomial of odd degree in  $n + 1$  variables, for  $i = 1, \dots, n$ , show that the equations  $f_i = 0$  have a common non-zero solution  $(x_0, \dots, x_n)$  in  $\mathbf{R}^{n+1}$ .

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