

Math 272a, homework 8

November 6, 2003

Problem 1. Show that \mathbf{RP}^n is orientable for n odd, and not orientable for n positive and even.

Problem 2. Analyse the proof of the existence of a fundamental class, and use a similar argument to show that $H_k(M)$ is zero for $k > n$ if M is a smooth manifold of dimension n . If M is also compact, prove that $H_k(M)$ is finitely generated for all k .

Problem 3. Let X be a space whose homology groups $H_k(X)$ are all finitely generated, with $H_k(X) = 0$ for $k \geq n$. The Euler number (or Euler characteristic) of X is then defined as

$$\chi(X) = \sum_{i=0}^n (-1)^i \text{rank } H_i(X).$$

If M is a compact orientable manifold of odd dimension, show that $\chi(M) = 0$. If M has dimension $4k + 2$ and is orientable, show that $\chi(M)$ is even. Do either of these statements remain true if M is not orientable?

Problem 4. Let $0 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 0$ be a short exact exact sequence of abelian groups. Show that such a sequence gives rise to long exact sequences

$$\cdots \rightarrow H_{i+1}(X; Q) \xrightarrow{\beta} H_i(X; H) \rightarrow H_i(X; G) \rightarrow H_i(X; Q) \rightarrow \cdots$$

and

$$\cdots \rightarrow H^{k-1}(X; Q) \xrightarrow{\beta'} H^k(X; H) \rightarrow H^k(X; G) \rightarrow H^k(X; Q) \rightarrow \cdots$$

Write down both of these sequences and determine the maps in them, in the case that $G = \mathbf{Z}$, $Q = \mathbf{Z}/2\mathbf{Z}$ and $X = \mathbf{RP}^3$.

Now suppose that X is a compact oriented n -manifold, and suppose $i + k = n$. Is there any relation between β and β' in this case?

Please also do questions 28 and 29 from section 3.3 of Hatcher's book.

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