Math 272a, homework 9

November 16, 2003

Problem 1. The *support* of a chain $\sum a_i \sigma_i \in C_*(X)$ is the union of the images of the singular simplices σ_i in X (assuming the coefficients a_i are non-zero and the σ_i are distinct). Let γ and γ' be a pair of 2-dimensional cycles in $\mathbf{CP}^2$. If the supports of γ and γ' are disjoint, show that at least one of γ and γ' must be a boundary.

Problem 2. Let M be a smooth compact n-manifold (without boundary). A *splitting* of M is a decomposition of M as $A \cup B$, where A and B are both n-manifolds with boundary, and $\partial A = \partial B = A \cap B$.

An *arbiter* (for splittings of M) is something (or someone) that does the following job. Given any splitting of M as $A \cup B$, the arbiter chooses one of the two pieces – the one the arbiter "likes". The arbiter's choices obey the following rules.

(a) The arbiter is *greedy*. If M is split as $A \cup B$ and the arbiter likes A, then the arbiter will definitely like A' if M is split as $A' \cup B'$ and $A' \supset A$.

(b) Let ϕ_t ($t \in [0, 1]$) be a continuous 1-parameter family of diffeomorphisms of M, with $\phi_0 = \mathrm{id}_M$. If the arbiter likes A in the splitting $A \cup B$, then the arbiter will definitely like $\phi_1(A)$ in the splitting $M = \phi_1(A) \cup \phi_1(B)$. This is *isotopy invariance*.

Show that there can be no arbiter for spittings of the Klein bottle. Show that there *is* an arbiter for splittings of the real projective plane, by describing one.

Remark. In addition to singular homology, a standard technical point may be useful in approaching this question: the $(n-1)$-manifold $N = A \cap B$ has neighborhood diffeomorphic to $[-1, 1] \times N$, in which N itself is embedded as $\{0\} \times N$.

Problem 3. Show that there is an arbiter for splittings of the complex projective plane. (There is more than one possible rule for the arbiter.)

The questions on the "arbiter" are due to Michael Freedman: at least, that is how they reached me. There can be no arbiter for splittings of M if the dimension of M is odd. (This is a non-trivial result.) Freedman reports he is unaware of any example of an arbiter that does not base its decisions on homology and does not use the axiom of choice.

Freedman is responsible for the proof of the topological Poincaré conjecture in dimension four: every topological 4-manifold with the homotopy type of S^4 is homeomorphic to S^4. It is not known whether there might be an exotic smooth *4-sphere: a smooth 4-manifold M that is homeomorphic but not diffeomorphic to S^4. If M were such an exotic 4-sphere, might there be an arbiter for M – an arbiter that likes to choose the piece in which the "exoticness" lies?*

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