Barycentric subdivision

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1. Subdivision of the standard simplex

Let $\Delta^n = [e_0, \ldots, e_n]$ be the standard n-simplex, the convex hull of $n + 1$ basis vectors $e_0, \ldots e_n$ in $\mathbb{R}^{n+1}$. For each bijective map

$$p : \{ 0, \ldots, n \} \to \{ 0, \ldots, n \}$$

and each j with $0 \le j \le n$, let $b_p^j \in \Delta^n$ be the point

$$b_p^j = \text{centroid of } \{ e_{p(j)}, \ldots, e_{p(n)} \}.$$

Let $\Delta_p^n \subset \Delta^n$ be the n-simplex

$$\Delta_p^n = [b_p^0, \ldots b_p^n].$$

We can write a typical point x of the simplex Δ^n as a linear combination

$$x = \sum_0^n x_i e_i,$$

where the x_i satisfy $0 \le x_i \le 1$ and $\sum x_i = 1$. The point x belongs to the simplex Δ_p^n if and only if

$$x_{p(0)} \le x_{p(1)} \le \cdots \le x_{p(n)}.$$

If the x_i are distinct, then x belongs to Δ_p^n for exactly one permutation p. This is the barycentric subdivision of the standard n-simplex.

2. The subdivision operator S

Let $\sigma : \Delta^n \to X$ be a singular simplex in the space X. Given an ordered $(k+1)$-tuple of points $(v_0, \ldots, v_k)$ in Δ^n, not necessarily distinct, let us write $\sigma[v_0, \ldots, v_k]$ as short-hand for the composite map

$$\sigma[v_0, \ldots, v_k] = \sigma \circ \theta : \Delta^k \to X,$$

where $\theta : \Delta^k \to \Delta^n$ is the linear map which sends e_i to v_i for $i = 0, \dots, k$. Define a homomorphism

$$S : C_n(X) \to C_n(X)$$

for all n, by specifying that, on a generator σ,

$$S\sigma = \sum_p (-1)^p \, \sigma[b_p^0, \dots, b_p^n].$$

Here the sum is over all permutations p, and $(-1)^p$ is notation for the sign of the permutation. Leaving out mention of the maps θ that are implicit in this definition, we can describe $S\sigma$ as the chain obtained by restricting σ to each of the n-simplices of the barycentric subdivision of Δ^n, and adding up the resulting singular simplices with signs.

The main result concerning S is that it is a chain map and is chain-homotopic to the identity. The latter is the statement that there is a homomorphism

$$K : C_n(X) \to C_{n+1}(X)$$

for all n, satisfying the identity

$$1 - S = \partial K + K\partial. \tag{1}$$

An identity of this sort actually implies that S is a chain map, i.e. that $\partial S = S\partial$: we have

$$\partial S - S\partial = -\partial(\partial K + K\partial) + (\partial K + K\partial)\partial,$$

which vanishes because $\partial\partial = 0$. Our main task is therefore to construct K and verify the identity (1).

3. Construction of the chain-homotopy *K*

The construction of K is based on a subdivision of the cylinder $[0, 1] \times \Delta^n$ into simplices. Although the definition of K and the verification of its properties are, in the end, entirely formal, we describe here the relevant subdivision of the cylinder. This subdivision equips the cylinder with the structure of a simplicial complex, in such a way that the top face $1 \times \Delta^n$ is triangulated by its barycentric subdivision, and the bottom face $0 \times \Delta^n$ is triangulated as a single n-simplex. A typical $n + 1$ simplex of the decomposition is the simplex $\Delta^{n+1}_{j,p}$ in $[0, 1] \times \Delta^n$ whose $n + 2$ vertices are the points

$$(1, b_p^0), \dots, (1, b_p^j), (0, e_{p(j)}), \dots, (0, e_{p(n)}.$$

Here $0 \le j \le n$, and p is a permutation which we can require to satisfy

$$p(j) < \cdots < p(n) \tag{2}$$

in order to avoid redundancy.

With this as motivation, we now define $K : C_n(X) \to C_{n+1}(X)$ by specifying that for a singular n-simplex $\sigma : \Delta^n \to X$ we have

$$K\sigma = \sum_j \sum_{p \in P(j)} (-1)^j (-1)^p \sigma[b_p^0, \ldots, b_p^j, e_{p(j)}, \ldots, e_{p(n)}]. \tag{3}$$

Here $P(j)$ is the set of permutations p satisfying the constraint (2). If we write $\pi : [0, 1] \times \Delta^n \to \Delta^n$ for the projection onto the second factor, we can think of $K\sigma$ as being a singular chain formed as a sum (with signs) of the restrictions of $(\sigma \circ \pi)$ to each of the simplices $\Delta_{j,p}^{n+1}$ that comprise the decomposition of $[0, 1] \times \Delta^n$:

$$K\sigma = \sum_j \sum_{p \in P(j)} (-1)^j (-1)^p (\sigma \circ \pi)|_{\Delta_{j,p}^{n+1}}.$$

In this less formal expression, the identification θ between the standard $(n + 1) - simplex$ and $\Delta_{j,p}^{n+1}$ is implied.

4. Verifying the chain-homotopy identity

We shall now prove the identity (1). For a singular k-simplex $\tau : \Delta^k \to X$, we write the boundary $\partial\tau \in C_{k-1}(X)$ as a sum

$$\partial\tau = \sum_{i=0}^{k} (-1)^i \partial^i \tau,$$

where

$$\partial^i \tau = \tau[e_0, \ldots, \widehat{e_i}, \ldots e_k].$$

With this convention, we write

$$\partial K\sigma = \sum_{j=0}^{n} \sum_{p \in P(j)} \sum_{i=0}^{n+1} (-1)^{i+j} (-1)^p \partial^i \left(\sigma[b_p^0, \ldots, b_p^j, e_{p(j)}, \ldots, e_{p(n)}]\right).$$

Gathering together terms of different types, we write

$$\partial K\sigma = A + B + C + D + E + F,$$

where the six parts are:

$$
\begin{array}{rl}
A: & \text{the terms with } i = 0 \text{ and } j = 0 \\
B: & \text{the terms with } i = n+1 \text{ and } j = n \\
C: & \text{the terms with } i = 0 \text{ and } j \geq 1 \\
D: & \text{the terms with } 1 \leq i \leq j-1 \\
E: & \text{the terms with } i = j \text{ and } j \geq 1 \\
F: & \text{the terms with } i > j \text{ and } j \leq n-1.
\end{array}
$$

We treat each of these in turn.

(i) The A terms

The set $P(j)$ consists of the identity permutation alone when $j = 0$, so there is only one term of type A, and we have

$$
\begin{aligned}
A &= \partial^0\big(\sigma[b^0_1, e_0, \ldots e_n]\big) \\
&= \sigma[\widehat{b^0_1}, e_0, \ldots e_n] \\
&= \sigma[e_0, \ldots e_n] \\
&= \sigma.
\end{aligned}
$$

(ii) The B terms

The set $P(n)$ is the whole permutation group, so the terms in B are

$$
\begin{aligned}
B &= \sum_p (-1)^{2n+1}(-1)^p \sigma[b^0_p, \ldots b^n_p, \widehat{e_{p(n)}}] \\
&= -\sum_p (-1)^p \sigma[b^0_p, \ldots b^n_p] \\
&= -S\sigma.
\end{aligned}
$$

(iii) The C terms

Blecchh …I did this in class, after a fashion. We have $C = -K\partial\sigma$.

(iv) The D terms

These terms cancel in pairs. A term in D has the form

$$
(-1)^j(-1)^p\sigma[b^0_p, \ldots, \widehat{b^i_p}, \ldots, b^j_p, e_{p(j)}, \ldots, e_{p(n)}],
$$

where $1 \leq i \leq j - 1$. This term should be matched against the term

$$(-1)^j (-1)^{p^*} \sigma [b^0_{p^*}, \dots, \widehat{b^i_{p^*}}, \dots, b^j_{p^*}, e_{p^*(j)}, \dots, e_{p^*(n)}],$$

where $p^* = p \circ t$ and t is the transposition $(i \;\, i-1)$. For $k \neq i$, the points b^k_p and $b^k_{p^*}$ coincide, so the simplices in the above two terms are identical. (Note that this would not be true if we allowed $i = j$, because in this case $e_{p^*(j)}$ and $e_{p(j)}$ would differ.) The signs of p and p^* are opposite, so the above two terms cancel. Thus $D = 0$.

(v) The E and F terms

Each E term is cancelled by one F term, so the sum $E + F$ is zero. Here is a typical F term:

$$(-1)^{i+j} (-1)^p \sigma [b^0_p, \dots, b^j_p, e_{p(j)}, \dots, \widehat{e_{p(i-1)}}, \dots, e_{p(n)}].$$

The permutation p belongs here to $P(j)$, and so it belongs a fortiori to the smaller set $P(j+1)$. Let p^* be the permutation in $P(j+1)$ given by $p^* = p \circ t$, where t is the cyclic permutation $t = (i-1 \;\; i-2 \; \dots \; j)$, so that

$$(p * (j+1), \dots, p^*(i-1)) = (p(j), \dots, p(i-2)).$$

The above F term can be matched with the following term of type E:

$$(-1)^{p^*} \sigma [b^0_{p^*}, \dots, \widehat{b^{j+1}_{p^*}}, e_{p^*(j+1)}, \dots, e_{p^*(n)}].$$

The vertices of the simplex in this E term coincide with those of the simplex in the F term. The signs of p and p^* differ by the sign of the permutation t, which is $(-1)^{i+j+1}$. So the above E and F terms cancel. In this way, the E and F terms pair off.

(vi) Conclusion of the calculation

We have shown that

$$\begin{aligned} \partial K \sigma &= A + B + C + D + (E + F) \\ &= \sigma - S\sigma - K\partial\sigma + 0 + 0. \end{aligned}$$

Thus $\partial K + K\partial = 1 - S$. □