A note about simple coverings of smooth manifolds by balls

For Math 272a, apropos of Poincaré duality

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For the purposes of these notes, we make the following definitions.

Definition 1. A closed subset U of an n-manifold X is a *standard ball* if there is a closed subset U^+ with $U \subset \operatorname{int}(U^+) \subset U^+$ such that the pair (U^+, U) is homeomorphic to the pair $(B^n, \frac{1}{2}B^n)$. The pair (U^+, U) is a *standard ball-pair*.

Definition 2. A *simple covering* of a manifold X is a finite collection of standard ball-pairs (U_i^+, U_i) $(i = 1, \dots, q)$, such that the interiors of the U_i cover X and such that for any

$$I = \{i_1, \dots, i_p\} \subset \{1, \dots, q\},$$

the pair

$$U_I^+ = U_{i_1}^+ \cap \cdots \cap U_{i_p}^+$$
$$U_I = U_{i_1} \cap \cdots \cap U_{i_p}$$

is either a standard ball-pair or a pair of empty sets.

Our aim is to prove:

Proposition 3. *Let X be a smooth, compact n-manifold. Then X has a simple covering.*

Proof. Embed X as a smooth submanifold of $\mathbb{R}^N$. Cover X by a finite collection of smooth charts $(\Omega_\alpha, \phi_\alpha)$: so Ω_α is open in $\mathbb{R}^n$ and

$$\phi_\alpha : \Omega_\alpha \to \mathbb{R}^N$$

is a diffeomorphism onto its image, an open subset of X. We can choose the coordinate charts so that for some positive constants $K_1 < K_2$, we have

$$K_1|x - y| \le |\phi_\alpha(x) - \phi_\alpha(y)| \le K_2|x - y|$$

for all α and all x, y in Ω_α (for example by taking the Ω_α to be bounded sets and arranging that ϕ_α extends to a smooth map on the closure of Ω_α). Similarly we can arrange that

$$\tau = \sup_{\alpha} \sup_{\substack{v \in T\Omega_\alpha \\ |v| \neq 0}} |\nabla_v \nabla_v \phi_\alpha| / |\nabla_v \phi_\alpha|^2$$

is finite. Finally, we can arrange that each Ω_α is convex.

Let ϵ_0 be a positive number sufficiently small that

(a) every subset of X of diameter no bigger than $4\epsilon_0$ is contained in $\phi_\alpha(\Omega_\alpha)$ for some α; and

(b) $(1 + K_2/K_1)\epsilon_0 < 1/\tau$.

Lemma 4. *Let $\epsilon \le \epsilon_0$ and let $B_1, \ldots, B_p$ be closed euclidean balls in $\mathbb{R}^N$ of radius ϵ about points $y_1, \ldots, y_p$. Suppose the intersection*

$$B_1 \cap \cdots \cap B_p \cap X$$

is non-empty. Then there exists α such that the interesection is the image $\phi_\alpha(C)$ of a compact, strictly convex subset $C \subset \Omega_\alpha$.

Proof. The first condition on ϵ_0 ensures that there exists an α such that $\phi_\alpha(\Omega_\alpha)$ contains each $B_i \cap X$. Since the intersection of strictly convex sets is strictly convex, it is enough to treat the case $p = 1$. We write $B_1 = B$ and $y_1 = y$. Let $C = \phi_\alpha^{-1}(B \cap X)$. Suppose c, c' are two different points of C, and let $\gamma_0 : [a, b] \to C$ be the straight-line path joining them, with $|\dot\gamma_0| = 1$. We must show that $\gamma_0(a, b) \subset \mathrm{int}(C)$, or equivalently that the path $\gamma = \phi_\alpha \circ \gamma_0 : [a, b] \to \mathbb{R}^N$ has $\gamma(a, b) \subset \mathrm{int}(B)$. (Note that $\phi_\alpha \circ \gamma_0$ is defined, because Ω_α is convex.)

Because $\gamma(a)$ and $\gamma(b)$ are in $B = B_\epsilon(y)$, we will be done if we show that $|\gamma(t) - y|^2$ is a strictly convex function of t. Note that each point $\gamma_0(t)$ ($t \in [a, b]$) is distance at most $(1/K_1)\epsilon$ from one of the endpoints. So each $\gamma(t)$ is distance at most $(K_2/K_1)\epsilon$ from either $\gamma(a)$ or $\gamma(b)$. From the triangle inequality it follows that

$$|\gamma(t) - y| \le (1 + K_2/K_1)\epsilon, \quad a \le t \le b.$$

Now

$$\begin{aligned}
\frac{d^2}{dt^2}|\gamma(t) - y|^2 &= 2\frac{d}{dt}\langle\dot\gamma, \gamma - y\rangle \\
&= 2\langle\dot\gamma, \dot\gamma\rangle + 2\langle\ddot\gamma, \gamma - y\rangle \\
&\ge 2|\dot\gamma|^2 - 2|\ddot\gamma||\gamma - y| \\
&\ge 2|\dot\gamma|^2 - 2(1 + K_2/K_1)\epsilon|\ddot\gamma| \\
&= 2|\nabla_{\dot\gamma_0}\phi_\alpha|^2 - 2(1 + K_2/K_1)\epsilon|\nabla_{\dot\gamma_0}\nabla_{\dot\gamma_0}\phi_\alpha| \\
&\ge 2|\nabla_{\dot\gamma_0}\phi_\alpha|^2\big(1 - (1 + K_2/K_1)\epsilon\tau\big) \\
&> 0,
\end{aligned}$$

which establishes the strict convexity. ☐

Now choose ϵ_1, smaller than ϵ_0, and finitely many ϵ_1-balls $B_1^+, \ldots, B_q^+$ whose interiors cover X. Let $U_i^+ = B_i \cap X$, and define U_I^+ as in Definition **??**. By the lemma, in some coordinate chart, U_I^+ is strictly convex or empty. If non-empty, it is therefore either a closed ball or a point. If any U_I^+ is a point, we may replace ϵ_1 by a slightly smaller value, so that this set becomes empty, while not creating any new single-point intersections. Thus we can ensure that each non-empty intersection is a ball. Choose

$\epsilon_2 < \epsilon_1$, and let U_i be the intersection of X with the ball B_i having radius ϵ_2 and the same center as B_i^+. Each non-empty intersection U_I^+ has non-empty interior, so by choosing ϵ_2 sufficiently close to ϵ_1, we can arrange that the corresponding intersection U_I has non-empty interior also. At the same time, we can ensure that the union of the interiors of the U_i is still all of X. At this point, we are done: U_I is contained in the interior of U_I^+, and the lemma tells us that each non-emtpy pair (U_I^+, U_I) is a standard ball-pair, because its inverse image under some coordinate chart ϕ_α is a pair of compact, strictly convex sets with non-empty interiors in $\mathbb{R}^n$. □