

Math 272b, sheet 1

February 9, 2004

Problem 1. Show that the tautological bundle on $\mathbf{RP}^1 \cong S^1$ is isomorphic to what we called the Möbius bundle.

Problem 2. If $M \rightarrow S^1$ is the Möbius bundle, show that $M \oplus M \cong \underline{\mathbf{R}} \oplus \underline{\mathbf{R}}$.

Problem 3. Show that TS^3 is trivial.

Problem 4. Show that $TS^2 \otimes TS^2$ is a trivial bundle of rank 4 on S^2 .

Problem 5. Let $L \rightarrow \mathbf{CP}^\infty$ be the tautological bundle. For $i = 1, 2$, let

$$L_i \rightarrow \mathbf{CP}^\infty \times \mathbf{CP}^\infty$$

be $\pi_i^*(L)$, where π_i is the projection on the i th factor. Since $L \rightarrow \mathbf{CP}^\infty$ is a universal bundle, there exists a map

$$f : \mathbf{CP}^\infty \times \mathbf{CP}^\infty \rightarrow \mathbf{CP}^\infty$$

such that $f^*(L) \cong L_1 \otimes L_2$. Describe such an f explicitly.

Problem 6. Let $U \subset (\mathbf{R}^\infty)^n$ be the set of linearly-independent n -tuples in the vector space \mathbf{R}^∞ . Show that U is contractible by describing an explicit contraction. Show that U is the total space of the frame bundle of the tautological bundle over the Grassmannian $G_n(\mathbf{R}^\infty)$. Deduce that $\pi_m(G_n(\mathbf{R}^\infty)) \cong \pi_{m-1}(GL(n, \mathbf{R}))$ for $m \geq 1$.

Problem 7. Starting from the previous problem, show that for every compact Lie group G , there is a principal G -bundle $p : P \rightarrow B$ whose base space B is paracompact and whose total space P is contractible. Use the fact that any compact Lie group is a subgroup of some orthogonal group.

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