Math 272b, homework 2

February 17, 2004

Problem 1. Let X be a smooth n-dimensional manifold admitting an immersion $f : X \to \mathbf{R}^{n+1}$. Let $w = 1 + w_1 + \cdots + w_n$ be the total Stiefel-Whitney class of X. Show that $(1 + w_1)w = 1$ in $H^2(X; \mathbf{Z}/2)$.

If X is a smooth n-dimensional manifold admitting an immersion $f : X \to \mathbf{R}^{n+2}$, show that $(1 + w_1 + w_1^2 - w_2)w = 1$.

The real projective plane $\mathbf{RP}^2$ admits an immersion in $\mathbf{R}^3$ as *Boy's surface*. It is hard to draw – here is a link to a picture of it. Use the existence of Boy's surface to provide a calculation of $w_2(\mathbf{RP}^2)$.

A smooth map $f : X \to Y$ between smooth manifolds is an ***immersion*** *if the derivative $f_* : TX \to TY$ is injective on each fiber T_xX, $x \in X$. This is equivalent to saying that each $x \in X$ has a neighborhood U_x such that the restriction of f embeds U_x as a smooth submanifold of Y.*

Problem 2. Show that $\mathbf{RP}^4$ does not admit an immersion in $\mathbf{R}^6$. (Note that $\mathbf{RP}^4$ is not orientable. It is true, in fact, that every *orientable* smooth 4-manifold can be immersed in $\mathbf{R}^6$.)

Problem 3. Let X be a smooth 3-manifold. Show that if the tangent bundle has $w_1 = 0$, then $w_2 = 0$ also.

Suggested strategy: Because $H^2(X; \mathbf{Z}/2) = \mathrm{Hom}(H_2(X; \mathbf{Z}/2), \mathbf{Z}/2)$, it is enough to check that the pairing of w_2 with every 2-dimensional $\mathbf{Z}/2$-homology class is zero. Now use the fact that every $\mathbf{Z}/2$-homology class in a 3-manifold can be represented as the fundamental class of a (not necessarily orientable) smooth 2-dimensional submanifold, $\Sigma \hookrightarrow X$. The previous exercises are then useful.

Problem 4. Use the fact that $\pi_4(S^3) = \mathbf{Z}/2$ to give a classification of complex vector bundles of rank 2 on S^5.

Remark. We discussed the "clutching construction" in class. Read more about the clutching construction in Hatcher's "Vector bundles and K-theory" to justify your answer to this question.

If a complex vector bundle $E \to X$ has $c_i(E) = 0$ for all $i \geq 0$, does it follow that E is trivial?

Problem 5. Let $E \to X$ be a complex vector bundle of rank n over a simplicial complex X of dimension at most $2n - 1$. Prove that E has a section $s : X \to E$ that is nowhere zero.

Show that every complex vector bundle $E \to S^4$ of rank $r \geq 2$ has the form $\underline{\mathbf{C}}^{r-2} \oplus E'$, where E' has rank 2 and $\underline{\mathbf{C}}^{r-2}$ denotes the trivial bundle of rank $r - 2$.

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