Math 272b, sheet 3

February 29, 2004

Problem 1. Let X be a CW complex and A a subcomplex. (Actually, we could work with any paracompact X and a subset A possessing a neighborhood $U \supset A$ such that A was a deformation-retract of U.)

Consider a vector bundle $E \to X$ of rank n together with a trivialization τ of $E|_A$:

$$\tau : E|_A \to A \times \mathbf{R}^n \quad (\text{or } \mathbf{C}^n).$$

Two such pairs (E, τ) and (E', τ') are *isomorphic* if there is a bundle isomorphism $\phi : E \to E'$ such that $\tau = \tau' \circ \phi|_A$.

Let G_n be the Grassmannian which is the base of the universal bundle, $E_n \to G_n$. Let $g_0 \in G_n$ be a base-point, and let σ_0 be an isomorphism of the fiber $E_n|_{g_0}$ with $\mathbf{R}^n$ or $\mathbf{C}^n$.

Given a pair (E, τ) over (X, A) as above, show that there is a map

$$f : (X, A) \to (G_n, g_0)$$

such that (E, τ) is isomorphic to the pull-back of (E_n, σ_0). (Define the pull-back first.)

Show that this establishes a one-to-one correspondence between the isomorphism classes of pairs (E, τ) and the set of homotopy classes, $[(X, A), (G_n, g_0)]$.

Give a definition of the *relative* Stiefel-Whitney classes, or relative Chern classes, of a pair (E, τ) as a class in $H^*(X, A; \mathbf{Z}/2)$, or $H^*(X, A; \mathbf{Z})$.

Problem 2. Let X be a 2-dimensional CW complex and let $E \to X$ be a real vector bundle of rank n. Let X^1 denote the 1-skeleton of X, as usual. Show that $E|_{X^1}$ is trivial if and only if $w_1(E) = 0$.

Now suppose $w_1(E) = 0$ and that the rank n is at least 3. Choose a trivialization τ of $E|_{X^1}$:

$$\tau : E|_{X^1} \to X^1 \times \mathbf{R}^n.$$

Show that the trivialization τ of $E|_{X^1}$ extends to a trivialization of E on $X^2 = X$ if and only if $w_2(E, \tau) = 0 \in H^2(X, X^1; \mathbf{Z}/2)$. (Deal with the case of a 2-disk first.)

Now suppose that τ' is a different trivialization of $E|_{X^1}$, agreeing with τ on X^0. Using the fact that $\pi_1(SO(n))$ is $\mathbf{Z}/2$, define *difference element*

$$\epsilon(\tau, \tau') \in H^1(X^1, X^0; \mathbf{Z}/2).$$

Show that

$$w_2(E, \tau') = w_2(E, \tau) + \partial\epsilon(\tau, \tau'),$$

where

$$\partial : H^1(X^1, X^0) \to H^2(X^2, X^1)$$

is the connecting homomorphism in the long exact sequence of the triple (X^2, X^1, X^0). (Again, this matter can be reduced to the case of the 2-disk.)

Deduce that E is trivial on $X = X^2$ if and only if $w_1(E)$ and $w_2(E)$ are both zero. Combine this with last week's homework, to deduce that the tangent bundle of an orientable 3-manifold is always trivial.

Problem 3. Suppose a compact, oriented manifold M admits an orientation-reversing self-diffeomorphism $f : M \to M$. Does it follow that the Pontryagin numbers of M are zero? Does it follow that the Euler number of M is zero?

Problem 4. Let $V \to \mathbf{RP}^2$ be the rank-2 vector bundle that arises as the normal bundle of the standard embedding $i : \mathbf{RP}^2 \hookrightarrow \mathbf{RP}^4$. This bundle is orientable. So, after choosing an orientation, there is a push-forward map $i_* : H^k(\mathbf{RP}^2) \to H^{k+2}(\mathbf{RP}^4)$ (with integer coefficients). Calculate i_* in the case $k = 2$.

Problem 5. Suppose a compact manifold M is the total space of a smooth fiber bundle $p : M \to B$ over a manifold B, with fiber F. (This means that that there is a covering of B by open sets U, and for each U a diffeomorphism $p^{-1}(U) \to U \times F$ commuting with the projections to U.) The *tangent space to the fibers* is the subbundle of TM whose fiber at m is $\mathrm{Ker}(Dp_m : T_m M \to T_{p(m)} B)$. Denote this subbundle by V (for 'vertical'). Show that $TM \cong V \oplus p^*(TB)$.

Now suppose that $M = \mathbf{P}(E)$, where $p : E \to B$ is a smooth complex vector bundle. Show that V has a complex structure and is isomorphic to $\mathrm{Hom}(L, L^\perp)$, where $L \to \mathbf{P}(E)$ is the tautological bundle and $p * (E) = L \oplus L^\perp$.

Finally, take the special case $M = \mathbf{P}(T\mathbf{CP}^2)$, where $T\mathbf{CP}^2$ denotes the tangent bundle viewed as a complex vector bundle. This M is a 6-manifold, and it has an almost complex structure because $TM = V \oplus p^*(T\mathbf{CP}^2)$ and both summands are complex. (In fact M is a complex manifold; it is the *flag manifold* of $\mathbf{C}^3$.) Calculate the Chern numbers $c_3[M]$, $c_1c_2[M]$ and $c_1^3[M]$.

Problem 6. *Optional.* Calculate the Chern numbers of the complex Grassmannian $M = G_2(\mathbf{C}^4)$. Calculate the Pontryagin numbers, $p_2[M]$ and $p_1^2[M]$ for this 8-manifold.

Very optional. Calculate the Pontryagin numbers of the 12-manifold $G_2(\mathbf{C}^5)$.

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