

Math 272b, homework 4

March 8, 2004

Problem 1. Prove or disprove: if $E \rightarrow X$ is a real oriented vector bundle over a finite cell complex, and if $e(E)$ is zero in $H^*(X; \mathbf{Z})$, then E has a section s which is nowhere zero.

Problem 2. Let $a_i \in \tilde{K}(S^n)$ be elements of reduced K -theory, ($i = 1, 2$). Explain how to join the a_i to obtain an element $a \in \tilde{K}(S^n \vee S^n)$. Let $f : S^n \rightarrow S^n \vee S^n$ be the standard map which in homology sends the generator 1 to $(1, 1)$. Show from the definitions that $f^*(a) = a_1 + a_2$ in $\tilde{K}(S^n)$. (“From the definitions” means that I don’t want you to use the fact that $K(S^n)$ is \mathbf{Z} for n even and 0 for n odd.)

Problem 3. Use your knowledge of vector bundles on S^2 and S^4 to show that an element of $K(\mathbf{CP}^2)$ is entirely determined by its Chern classes. Hence compute $K(\mathbf{CP}^2)$ and describe generators.

Problem 4. Recall that the standard inclusion of the unitary groups $U(N) \rightarrow U(N+1)$ gives an isomorphism $\pi_k(U(N)) \rightarrow \pi_k(U(N+1))$ for $k < 2N$, as one sees from the fibration

$$U(N) \rightarrow U(N+1) \rightarrow S^{2N+1}.$$

It follows that there is an isomorphism

$$i_* : \pi_k(GL_N(\mathbf{C})) \rightarrow \pi_k(GL_{2N}(\mathbf{C}))$$

for k in the same range. So Whitney sum defines a law of composition on $\pi_k(GL_N(\mathbf{C}))$: given two maps $A, B : S^k \rightarrow GL_N(\mathbf{C})$, let C be the homotopy class of the map

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : S^k \rightarrow GL_{2N}(\mathbf{C}),$$

and define $A \odot B$ to be the element $(i_*)^{-1}(C) \in GL_N(\mathbf{C})$. On the other hand, we can form the matrix product $AB : S^k \rightarrow GL_N(\mathbf{C})$, and then there is the homotopy class $A + B$ constructed using the usual definition of the group law on the homotopy groups. Show that $A \odot B$ and AB are equal in $\pi_k(GL_N(\mathbf{C}))$.

Remark. It is also a standard fact about topological groups that AB and $A + B$ are homotopic: I think this appeared on the homeworks last semester.

Problem 5. Let H be a separable complex Hilbert space and \mathcal{A} the algebra of bounded operators. It is a theorem of Kuiper's that the set of invertible elements \mathcal{A}^* is contractible.

Let e_i ($i \geq 0$) be a complete orthonormal system in H , and for $z \in S^1$ let T_z be the operator with $T_z e_0 = z e_0$ and $T_z e_i = e_i$ for $i > 0$.

Thus we have defined a loop $T : S^1 \rightarrow \mathcal{A}^*$. Give an explicit map $D^2 \rightarrow \mathcal{A}^*$ whose restriction to the circle is T , so showing that this loop is null-homotopic.

Remark. If you can't find a solution, search for Kuiper's original paper on the contractibility of \mathcal{A}^* (in which the contractibility of the Hilbert unitary group is also established).