

Math 272b, homework 6

March 20, 2004

Problem 1. Calculate the reduced K -theory of \mathbf{RP}^n for small n ($n = 2, 3$ and 4) by using the long-exact sequence of the pair $(\mathbf{RP}^n, \mathbf{RP}^{n-1})$. Since $\mathbf{RP}^n/\mathbf{RP}^{n-1}$ is S^n , the long exact sequence looks like:

$$\begin{aligned} &\rightarrow \mathbf{Z} \rightarrow \tilde{K}^{-1}(\mathbf{RP}^n) \rightarrow \tilde{K}^{-1}(\mathbf{RP}^{n-1}) \\ &\rightarrow 0 \rightarrow \tilde{K}^0(\mathbf{RP}^n) \rightarrow \tilde{K}^0(\mathbf{RP}^{n-1}) \\ &\rightarrow \mathbf{Z} \rightarrow \dots \end{aligned}$$

for n odd, and

$$\begin{aligned} &\rightarrow 0 \rightarrow \tilde{K}^{-1}(\mathbf{RP}^n) \rightarrow \tilde{K}^{-1}(\mathbf{RP}^{n-1}) \\ &\rightarrow \mathbf{Z} \rightarrow \tilde{K}^0(\mathbf{RP}^n) \rightarrow \tilde{K}^0(\mathbf{RP}^{n-1}) \\ &\rightarrow 0 \rightarrow \dots \end{aligned}$$

for n even. Use your knowledge of vector bundles and characteristic classes to try and resolve any ambiguities that arise in determining the groups.

To get started, consider $n = 2$. In the long exact sequence, there is a map

$$\delta : \tilde{K}^{-1}(\mathbf{RP}^{n-1}) \rightarrow \mathbf{Z} = \tilde{K}^0(\mathbf{RP}^2/\mathbf{RP}^1),$$

i.e. $\delta : \mathbf{Z} \rightarrow \mathbf{Z}$. To determine δ , we can use the commutative diagram

$$\begin{array}{ccc} S^1 & \longrightarrow & D^2 \\ \downarrow & & \downarrow \\ \mathbf{RP}^1 & \longrightarrow & \mathbf{RP}^2. \end{array}$$

When you get to $n = 4$, you will want to show that there is an element x in $\tilde{K}(\mathbf{RP}^4)$ with $2x \neq 0$.

Remark. This is not the best way to set about calculating the K -theory of \mathbf{RP}^n in general.

Problem 2. In the context of ordinary cohomology, we proved the existence and uniqueness of the Thom class for orientable vector bundles. Do an analysis of the proof to see where the result fails for K -theory.

In more detail, let $E \rightarrow X$ be an oriented vector bundle, say of even rank. Take X to be a finite CW complex, for example. For each $x \in X$, we have $K(E_x) = \mathbf{Z}$, and the orientation of E_x picks out a preferred generator $\mu_x \in K(E_x)$ for all x . A Thom class would be a class $u \in K(E)$ such that $j_x^*(u) = \mu_x$ for all x (where $j_x : E_x \rightarrow E$ is the inclusion). Why does the proof of existence for ordinary cohomology not work for K -theory?

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