Math 272b, homework 7

April 6, 2004

Problem 1. Find an expression for the Todd class of $\mathbf{CP}^n$, as a series in the generator $h \in H^2(\mathbf{CP}^n)$.

Remark. Do not seek a closed form for the individual coefficients, only for the whole. For example, an appropriate (but incorrect) answer might be Todd $= \sin h$.

Problem 2. Let E be a vector bundle on $\mathbf{CP}^4$ having $c_i(E) = 0$ for $i = 1, 3, 4$. We can write $c_2(E) = ah^2$ for some integer a. Prove that $a^2 + a$ must be divisible by 12.

Give two proofs: one using your knowledge of the explicit generators of $K^*(\mathbf{CP}^n)$, and another using the integrality theorem for $\mathrm{ch}(E)\mathrm{Todd}(X)[X]$.

When X and Y are smooth compact manifolds equipped with stable almost complex structures, and $f : X \to Y$ is a map, we defined in class a push-forward map

$$f_* : K^*(X) \to K^*(Y).$$

Recall that the definition involved choosing an embedding $i : X \to \mathbf{R}^n$ for some large n, and then applying the K-theory Thom isomorphism theorem to the normal bundle of the embedding $(i \times f) : X \to \mathbf{R}^n \times Y$.

Problem 3. Let X be a point and Y a compact manifold of dimension d with a stable almost complex structure. Let $f : X \to Y$ be a map, and let Y_1 be the component of Y containing $f(X)$. Let $g : Y \to S^d$ be a map whose restriction to Y_1 and has degree 1 and whose restriction to the other components of Y is constant. Let $b \in \tilde{K}^*(S^d)$ be the generator, and let $a = g^*(b)$. Show that $f_*(1) = \pm a$.

Problem 4. Let $f : \mathbf{CP}^2 \to \mathbf{CP}^4$ be the inclusion. Calculate the push-forward map $f_* : H^*(\mathbf{CP}^2) \to H^*(\mathbf{CP}^4)$. (Use Poincaré duality, perhaps.)

Now use the Chern character and your knowledge of the Todd class of $\mathbf{CP}^n$ to calculate the map $f_* : K(\mathbf{CP}^2) \to K(\mathbf{CP}^4)$.

Remark. When I say "calculate" here, I mean describe f_* in terms of the standard generators $1, x, \dots, x^n$ for $K(\mathbf{CP}^n)$, where $x = [H] - 1$.

Problem 5. *Optional*. Recall from class that the *Todd genus* of a compact almost complex manifold X is the integer Todd$[X]$ obtained by evaluating the Todd class of X on the fundametal class $[X]$. For example, if the real dimension of X is 4, then

$$\mathrm{Todd}[X] = \frac{1}{12}(c_1^2 + c_2)[X]$$

Show that the Todd genus of $\mathbf{CP}^n$ is 1, for all n.

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