

## Math 272b, homework 9

*April 25, 2004*

**Problem 1.** Show that the circle  $S^1$  has two distinct spin structures. For each spin structure, calculate  $f_*(1)$ , where

$$f : KO^0(S^1) \rightarrow KO^{-1}(\text{point}) = \mathbf{Z}/2.$$

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**Problem 2.** Calculate  $MSO_k(\mathbf{RP}^\infty)$  for  $k \leq 3$ , and describe explicit generators. Exhibit a map  $MO_3(\mathbf{RP}^\infty) \rightarrow \mathbf{Z}/2 \oplus \mathbf{Z}/2$ , and show your map is surjective.

Hint: a map to  $\mathbf{RP}^\infty$  is the same as giving a real line bundle, or an element of  $H^1(X; \mathbf{Z}/2)$ .

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**Problem 3. *Optional.*** If  $E$  is a real vector bundle, let  $\mathbf{RP}(E)$  denote the projectivization, a fiber bundle with fibers real projective spaces.

Let  $M$  be a smooth compact manifold, with an involution  $\tau$ , i.e. a smooth self-diffeomorphism of order 2. Let  $F$  be the fixed point set. You may assume the basic differential-topological results:  $F$  must be a smooth submanifold; the derivative of  $\tau$  induces the map  $-1$  on  $\nu(F)$ ; and there is a  $\mathbf{Z}/2$ -equivariant version of the tubular neighborhood theorem – there is a tubular neighborhood of  $F$  which can be identified with the ball bundle of  $\nu(F)$  in a way that respects the action of the involution. (If you ever proved the tubular neighborhood theorem by introducing a riemannian metric and using the exponential map, then you can prove this equivariant version too, by introducing an invariant riemannian metric.)

With that background, prove that  $[M] = [\mathbf{RP}(\nu(F) \oplus \mathbf{R})]$  in the unoriented cobordism group  $\mathrm{MO}_*(\mathrm{pt.})$ , by exhibiting a cobordism.

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**Problem 4. *Optional.*** Use the above problem to show that  $[\mathbf{RP}^n \times \mathbf{RP}^n] = [\mathbf{CP}^n]$  in  $\mathrm{MO}_{2n}(\mathrm{pt.})$ .

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**Problem 5. *Optional.*** Prove that if  $M$  is any compact manifold, then  $\mathbf{RP}(TM \oplus \mathbf{R})$  is cobordant to  $M \times M$ . (We are still talking about unoriented cobordism.)

Deduce that if  $M$  is an odd-dimensional manifold, then  $M \times M$  is cobordant to an orientable manifold. (The same is true in the even-dimensional case, but this result needs more input.)

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