

Bott periodicity and harmonic theory on the 3-sphere

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1. Harmonic theory on the 3-sphere

(i) The spherical Laplacian

We use the geometer's Laplacian, which is *minus* the sum of the second derivatives: on \mathbb{R}^n with coordinates (x_0, \dots, x_{n-1}) , the operator is

$$\Delta_{\mathbb{R}^n} f = - \sum_{b=0}^{n-1} \frac{\partial^2}{\partial x_b^2} f.$$

If r is the radial distance in \mathbb{R}^n , then the Laplacian can be expressed as

$$\Delta_{\mathbb{R}^n} f = - \frac{\partial^2}{\partial r^2} f - \frac{n-1}{r} \frac{\partial}{\partial r} f + \frac{1}{r^2} \Delta_{S^{n-1}} f, \quad (1)$$

where $\Delta_{S^{n-1}}$ is an operator on functions on the unit sphere: the spherical Laplacian.

In the case of \mathbb{R}^4 , we identify \mathbb{R}^4 with the unit quaternions \mathbb{H} , with real basis

$$(1, i, j, k) = (1, I_1, I_2, I_3).$$

For each point z on the unit sphere $S^3 \subset \mathbb{H}$, we have three orthogonal vectors, in the directions $I_a z$, ($a = 1, 2, 3$). These define three mutually orthogonal tangent vector fields on the 3-sphere. We denote by ∂_a the operator on smooth, real-valued functions,

$$\partial_a : C^\infty(S^3) \rightarrow C^\infty(S^3)$$

given by differentiating along the vector field $I_a z$ on S^3 : so

$$(\partial_a h)(z) = \frac{d}{d\theta} h(e^{I_a \theta} z)|_{\theta=0}.$$

From the fact that the vector field defining ∂_a is volume-preserving, it follows that

$$\int_{S^3} \partial_a f \, d\text{vol} = 0,$$

and from the Leibnitz rule we then obtain that ∂_a is skew-adjoint:

$$\langle \partial h_1, h_2 \rangle = -\langle h_1, \partial h_2 \rangle,$$

where

$$\langle h_1, h_2 \rangle = \int_{S^3} h_1 h_2 \, d\text{vol}.$$

The spherical Laplacian can be expressed as

$$\Delta_{S^3} = - \sum_{a=1}^3 \partial_a^2.$$

This operator is self-adjoint.

(ii) Spectrum of the spherical Laplacian

We say that a smooth function f on S^3 is an eigenvector of the Laplacian if

$$\Delta_{S^3} f = \lambda f$$

for some λ . Because the Laplacian is self-adjoint, the eigenvectors belonging to different eigenvalues are orthogonal.

Theorem 1. *The real Hilbert space $L^2(S^3)$ has a complete orthonormal system consisting of eigenvectors of the Laplacian. The eigenvalues of Δ_{S^3} are the integers*

$$\lambda_d = (d + 1)^2 - 1, \quad (d \geq 0).$$

The dimension of the eigenspace F_d belonging to λ_d is $(d + 1)^2$. Furthermore, the subspace

$$F_0 \oplus F_1 \oplus \cdots \oplus F_d \subset C^\infty(S^3)$$

is precisely the space of functions on S^3 that are the restrictions to S^3 of polynomial functions on \mathbb{R}^4 of degree at most d .

The theorem is stated for the 3-sphere only, but (with different eigenvalues and multiplicities), the result is general. The proof involves the polynomial functions $p(x_0, \dots, x_{n-1})$ on \mathbb{R}^n that are harmonic, i.e. that satisfy $\Delta_{\mathbb{R}^n} p = 0$.

Proposition 2. *For every polynomial function q on \mathbb{R}^n , there is a unique harmonic polynomial p whose restriction to the unit sphere is equal to the restriction of q . Furthermore, the degree of p is not larger than the degree of q .*

Proof. Let Q_d denote the linear space of real-valued polynomial functions on \mathbb{R}^n of degree at most d . Let R denote the map obtained by restricting to the sphere:

$$R : Q_d \rightarrow C^\infty(S^3).$$

We also have the Laplace operator, whose kernel is the space of harmonic polynomials. It lowers degree by 2:

$$\Delta_{\mathbb{R}^n} : Q_d \rightarrow Q_{d-2}.$$

The maximum principle for harmonic functions tells us that a harmonic function that vanishes on the unit sphere must vanish in the ball. So in Q_d ,

$$\ker(R) \cap \ker(\Delta) = 0.$$

We can identify $\ker(R)$ as the polynomials divisible by $(r^2 - 1)$, the subspace $(r^2 - 1)Q_{d-2} \subset Q_d$. We therefore have

$$\dim \ker(R) = \dim Q_{d-2}.$$

On the other hand

$$\begin{aligned} \dim \ker(\Delta) &= \dim Q_d - \text{rank } \Delta \\ &\geq \dim Q_d - \dim Q_{d-2}. \end{aligned}$$

Because the intersections of the kernels is zero, it now follows from examining dimensions that

$$Q_d = \ker(R) \oplus \ker(\Delta).$$

This is equivalent to the statement of the Proposition. \square

Before moving on, we note a corollary of the dimension-counting in the proof above:

Corollary 3. *Acting on polynomials, the Laplace operator $\Delta_{\mathbb{R}^n} : Q_d \rightarrow Q_{d-2}$ is surjective.* \square

Now let P_d be the space of polynomials that are homogeneous of degree d . Let $H_d \subset P_d$ denote the space of homogeneous harmonic polynomials. The homogeneous parts of a harmonic polynomial are harmonic, so the space of all harmonic polynomials is the direct sum of the spaces H_d . The Proposition tells us that

$$\dim H_d = \dim P_d - \dim P_{d-2}.$$

In dimension n , we have

$$\dim P_d = \binom{d+n-1}{n-1}.$$

So in dimension $n = 4$ we have:

$$\begin{aligned} \dim H_d &= \frac{1}{6}((d+3)(d+2)(d+1) - (d+1)(d)(d-1)) \\ &= (d+1)^2. \end{aligned} \tag{2}$$

Lemma 4. *If $f \in H_d$ is a harmonic polynomial on \mathbb{R}^n of degree d , then its restriction to the unit sphere S^{n-1} is an eigenvalue of the Laplacian $\Delta_{S^{n-1}}$ with eigenvalue $d^2 + (n-2)d$.*

Proof. Because f is homogeneous of degree d , we have

$$\frac{\partial}{\partial r} f = \frac{d}{r} f.$$

If we write out the condition $\Delta_{\mathbb{R}^n} f = 0$ using the expression (1) for the Laplacian, we obtain

$$\begin{aligned} \Delta_{S^{n-1}} f &= d(d-1)f + (n-1)df \\ &= (d^2 + (n-2)d)f. \end{aligned}$$

□

Proof of Theorem 1. Let $F_d \subset C^\infty(S^3)$ denote the space of functions obtained as the restriction of harmonic polynomials belonging to H_d . By the lemma above, F_d is contained in the eigenspace belonging (when $n = 3$) to the eigenvalue $d^2 + 2d = (d+1)^2 - 1$, and its dimension is $(d+1)^2$. Furthermore, the last statement of the Theorem holds for these subspaces F_d , by Proposition 1(ii). We must show that orthonormal bases for the F_d , taken together, are a complete orthonormal system in $L^2(S^3)$. As remarked earlier, because the eigenvalues are distinct, the F_d are mutually orthogonal. The remaining point is that the sum of all the F_d is dense in $L^2(S^3)$, or equivalently that the functions obtained by restricting arbitrary polynomials are dense. In this last form, the assertion follows from the Weierstrass approximation theorem, which states that any continuous function on a compact set in \mathbb{R}^n can be uniformly approximated by polynomials. □

Examples. The homogeneous harmonic polynomials of degree 0 are the constants. They restrict to the sphere S^3 as the constant functions F_0 . All linear polynomials are harmonic: the space F_1 is 4-dimensional, and consists of the restrictions of the linear functions. The corresponding eigenvalue is 3. A basis for the homogeneous, quadratic harmonic functions on \mathbb{R}^4 is

$$x_0x_1, x_0x_2, x_0x_3, x_1x_2, x_1x_3, x_2x_3, x_0^2 - x_1^2, x_1^2 - x_2^2, x_2^2 - x_3^2.$$

These span the 9-dimensional eigenspace belonging to the eigenvalue 8.

(iii) A Dirac-type operator

Let $C^\infty(S^3; \mathbb{H})$ be the space of smooth quaternion-valued functions on the unit sphere. Quaternions act on this space, by multiplication on both the left and

right. We will regard this space as a left-quaternion vector space: i.e. a left module over \mathbb{H} . Define an operator

$$D : C^\infty(S^3; \mathbb{H}) \rightarrow C^\infty(S^3; \mathbb{H})$$

by

$$Df(z) = - \sum_{a=1}^3 \partial_a f(z) I_a.$$

Like the Laplacian, this operator is self-adjoint: that is, we have

$$\langle Dh_1, h_2 \rangle = \langle h_1, Dh_2 \rangle, \quad (3)$$

where the inner product here is the real inner product

$$\langle h_1, h_2 \rangle = \int_{S^3} \operatorname{Re}(h_1 \bar{h}_2) \, d\operatorname{vol}. \quad (4)$$

(Inside the parentheses is the quaternion product of h_1 and the conjugate of h_2 .) The identity (3) follows from the fact that the operators $h \mapsto \partial_a h$ and $h \mapsto h I_a$ commute and are both skew-adjoint. Note also that D is a homomorphism of left \mathbb{H} -modules:

$$D(ch) = cDh, \quad (c \in \mathbb{H}).$$

Examples. The constant functions are in the kernel of D :

$$Dc = 0, \quad (c \in \mathbb{H}). \quad (5)$$

If we regard S^3 again as sitting in \mathbb{H} , then the function $f(z) = z$ is an \mathbb{H} -valued function on S^3 . Denoting this function simply by z , we compute

$$\begin{aligned} Dz &= - \sum \frac{d}{d\theta} (e^{I_a \theta} z)|_{\theta=0} I_a \\ &= - \sum I_a z I_a \\ &= z + 2\bar{z}. \end{aligned}$$

Next, we compute D applied to the function $f(z) = \bar{z}$:

$$\begin{aligned} D\bar{z} &= - \sum \frac{d}{d\theta} (\overline{e^{I_a \theta} z})|_{\theta=0} I_a \\ &= - \sum \frac{d}{d\theta} (\bar{z} e^{-I_a \theta})|_{\theta=0} I_a \\ &= \sum \bar{z} I_a^2 \\ &= -3\bar{z}. \end{aligned}$$

Combining the last two calculations, we obtain

$$D(2z + \bar{z}) = 2z + \bar{z}.$$

Thus 0, -3 and 1 are all eigenvalues of D . We record one of these examples for future reference:

Lemma 5. *The functions $f(z) = c\bar{z}$ ($c \in \mathbb{H}$) are eigenfunctions of D belonging to the eigenvalue -3 . \square*

The operator D is closely related to the standard Dirac operator on the 3-sphere: the usual Dirac operator \mathcal{D} is equivalent to $D + 3/2$. Our presentation has hidden some of the symmetry, so that the proof of the next lemma seems somewhat ad hoc. In terms of the standard Dirac operator, the lemma says that \mathcal{D} is unitarily equivalent to $-\mathcal{D}$.

Lemma 6. *The operators $D + 3$ and $-D$ are unitarily equivalent: there is a norm-preserving automorphism $\dagger : C^\infty(S^3; \mathbb{H}) \rightarrow C^\infty(S^3; \mathbb{H})$ such that*

$$Df^\dagger + 3f^\dagger = (-Df)^\dagger.$$

Proof. Define \dagger by

$$f^\dagger(z) = f(\bar{z})\bar{z}.$$

Write $I_a^z = zI_a\bar{z}$, and let ∂_a^z be the vector field on S^3 given by the tangent vector $I_a^z z$ at $z \in S^3$: that is,

$$(\partial_a^z f)(z) = \frac{d}{d\theta} f(e^{I_a^z \theta} z)|_{\theta=0}.$$

By linearity of the derivative, we can express D also in terms of these new vector fields:

$$\begin{aligned} (Df)(z) &= - \sum_a \partial_a^z f(z) I_a^z \\ &= - \sum_a \frac{d}{d\theta} f(e^{zI_a\bar{z}\theta} z) (zI_a\bar{z}) \\ &= - \sum_a \frac{d}{d\theta} f(e^{I_a\theta} z) zI_a\bar{z}. \end{aligned}$$

Now apply this expression for D to f^\dagger :

$$\begin{aligned} (Df^\dagger)(z) &= - \sum_a \frac{d}{d\theta} (f(e^{-I_a\theta}\bar{z})e^{-I_a\theta}\bar{z}) zI_a\bar{z} \\ &= - \sum_a \frac{d}{d\theta} f(e^{-I_a\theta}\bar{z}) I_a\bar{z} - \sum_a f(\bar{z}) (-I_a^2)\bar{z} \\ &= \sum_a \frac{d}{d\theta} f(e^{I_a\theta}\bar{z}) I_a\bar{z} - \sum_a f(\bar{z})\bar{z} \\ &= (-Df)^\dagger(z) - 3f^\dagger(z). \end{aligned}$$

This gives the result. \square

(iv) Spectrum of the 3-dimensional Dirac operator

We regard the quaternion-valued, square-integrable functions $L^2(S^3; \mathbb{H})$ as a real Hilbert space with the inner product (4).

Theorem 7. *The Hilbert real space $L^2(S^3; \mathbb{H})$ has a complete orthonormal system consisting of eigenvectors for the operator D . The eigenvalues are all the integers d (positive and negative) with the exception of $d = -1$ and $d = -2$. The dimension of the eigenspace belonging to the eigenvalue d is $2(d+1)(d+2)$.*

Proof. The proof is based on a relationship between the operator D and the spherical Laplacian. We continue to write Δ_{S^3} where we might more properly write $\Delta_{S^3} \otimes 1$, for the Laplace operator acting on \mathbb{H} -valued functions.

Lemma 8. *We have $(D+1)^2 = \Delta_{S^3} + 1$ as operators on $C^\infty(S^3; \mathbb{H})$.*

Proof. Let (a, b, c) be a cyclic permutation of $(1, 2, 3)$, so that $I_a I_b = -I_b I_a = I_c$. We then have

$$\partial_a \partial_b - \partial_b \partial_a = -2\partial_c,$$

as we can verify from the definition:

$$\begin{aligned} (\partial_a \partial_b - \partial_b \partial_a)f(z) &= \frac{d}{d\theta_1} \frac{d}{d\theta_2} (f(e^{I_b \theta_1} e^{I_a \theta_2} z) - f(e^{I_a \theta_1} e^{I_b \theta_2} z))|_{\theta_1=\theta_2=0} \\ &= \frac{d}{d\theta_1} \frac{d}{d\theta_2} f(z + \theta_1 \theta_2 (I_b I_a - I_a I_b)z) \\ &= -2\partial_c f(z). \end{aligned}$$

Using this relation, we compute

$$\begin{aligned} D^2 f(z) &= -\sum_a \partial_a^2 f(z) + \sum_{a \neq b} \partial_a \partial_b f(z) I_b I_a \\ &= -\sum_a \partial_a^2 f(z) + \sum_{a < b} (\partial_a \partial_b - \partial_b \partial_a) f(z) I_b I_a \\ &= -\sum_a \partial_a^2 f(z) + 2 \sum_c (\partial_c) f(z) I_c \\ &= \Delta_{S^3} f(z) - 2Df(z). \end{aligned}$$

Thus $D^2 = \Delta_{S^3} - 2D$, which is equivalent to the assertion of the Lemma. \square

From Theorem 1, the subspace $F_d \otimes \mathbb{H} \subset C^\infty(S^3; \mathbb{H})$ is an eigenspace for Δ_{S^3} belonging to the eigenvalue $\lambda_d = (d+1)^2 - 1$. From the Lemma, it follows that D and Δ_{S^3} commute, so $F_d \otimes \mathbb{H}$ is invariant under D . Because the restriction of D to this finite-dimensional inner product space is self-adjoint, $F_d \otimes \mathbb{H}$ has an orthonormal basis consisting of eigenvectors of D . The sum of these subspaces is dense in $L^2(S^3; \mathbb{H})$, so the Hilbert space $L^2(S^3; \mathbb{H})$ admits a complete orthonormal system of eigenvectors of D .

To find the eigenvalues, we note that an eigenvalue ϵ of D in $F_d \otimes \mathbb{H}$ satisfies

$$\begin{aligned} (\epsilon + 1)^2 &= \lambda_d + 1 \\ &= (d + 1)^2 \end{aligned}$$

by the Lemma. So $\epsilon = d$ or $-d - 2$. In particular, since F_d is non-zero only for $d \geq 0$, the eigenvalues of D are contained in $\mathbb{Z} \setminus \{-1\}$.

Let E_d be the eigenspace belonging to the eigenvalue $\epsilon = d$. We have seen the relation

$$E_d \oplus E_{-2-d} = F_d \otimes \mathbb{H}, \quad (d \geq 0),$$

and from Lemma 6, we have

$$E_d \cong E_{-3-d}.$$

Combining these two, we have

$$\dim E_d + \dim E_{d-1} = 4 \dim F_d.$$

We know that $\dim E_{-1} = 0$ and $\dim F_d = (d + 1)^2$. The formula

$$\dim_{\mathbb{R}} E_d = 2(d + 1)(d + 2)$$

for all positive d follows readily by induction. For negative d , it then follows from the relation $\dim E_d = \dim E_{3-d}$. \square

2. Fredholm theory

(i) Sobolev spaces

Write $\mathcal{H} = L^2(S^3; \mathbb{H})$. We can regard this as a real Hilbert space, and as left module for the quaternions. From Theorem 7, we have a decomposition of \mathcal{H} as a Hilbert sum:

$$\mathcal{H} = \hat{\bigoplus}_d E_d.$$

We mean that the summands are mutually orthogonal, and that \mathcal{H} is the closure of their direct sum. We can also think of \mathcal{H} as the external Hilbert sum: the set of sequences

$$\left\{ u = (u_d)_{d \in \mathbb{Z}} \mid u_d \in E_d, \sum_d \|u_d\|^2 < \infty \right\}.$$

Either way, \mathcal{H} is the completion, with respect to the standard L^2 norm, of the space of polynomial functions on the sphere with values in \mathbb{H} : the space

$$\mathcal{E} = \bigoplus_d E_d.$$

We will also need a version of this Hilbert space in which the norm has been weighted. These are the Sobolev spaces. Because -1 is not in the spectrum of D , the operator $D + 1$ is invertible on \mathcal{E} . We write δ for the absolute value of this operator:

$$\delta = |D + 1|.$$

This is to say that the restriction of δ to E_d is multiplication by $|d + 1|$. For any real s , we define the Sobolev norm $\|u\|_s$ for $u \in \mathcal{E}$ by

$$\|u\|_s = \|\delta^s u\|.$$

The Sobolev space \mathcal{H}_s is defined as the Hilbert space obtained by completing \mathcal{E} using the Sobolev norm. Thus

$$\mathcal{H}_s = \left\{ u = (u_d)_{d \in \mathbb{Z}} \mid u_d \in E_d, \sum_d |d + 1|^{2s} \|u_d\|^2 < \infty \right\}.$$

The operator δ^t on \mathcal{E} extends to an isometry from \mathcal{H}_s to \mathcal{H}_{s-t} .

Proposition 9. *Let f and u belong to \mathcal{E} , and let fu be their quaternion product. Then for all s in the interval $-1 \leq s \leq 1$, we have*

$$\|fu\|_s \leq \|f\|_{C^1} \|u\|_s,$$

where the C^1 norm of f is the quantity

$$\|f\|_{C^1} = \sup_z |f(z)| + \sum_a \sup_z |\partial_a f(z)|.$$

Proof. First we verify the inequality for $s = 1$. In this case, $\|fu\|_1 = \|(D + 1)fu\|$. We calculate

$$(D + 1)fu = f(D + 1)u - \sum_a (\partial_a f)uI_a.$$

The L^2 norm of the first term is bounded by

$$\left(\sup_z |f(z)| \right) \|u\|_1.$$

The L^2 norm of the a th term in the sum is bounded by

$$\left(\sup_z |\partial_a f(z)| \right) \|u\|_0.$$

So the result follows in this case.

For general s , write the inequality to be proved as

$$|\langle \delta^s fu, \delta^s w \rangle| \leq \|f\|_{C^1} \|\delta^s u\| \|\delta^s w\|$$

for u and w in \mathcal{E} . Write $x = \delta^s u$ and $y = \delta^s w$. Then the inequality becomes

$$|\langle \delta^s f \delta^{-s} x, y \rangle| \leq \|f\|_{\mathcal{C}^1} \|x\| \|y\|.$$

for all x and y in \mathcal{E} . Here the norms are the ordinary L^2 norms. In this formulation, one sees by taking adjoints that if the inequality holds for a given s and a given f , then it also holds for $-s$ and \bar{f} . In particular, the Proposition holds for $s = -1$ and all f .

For s between -1 and 1 , we can use an interpolation argument. For fixed polynomials f , x and y , the function

$$v(s) = \langle \delta^s f \delta^{-s} x, y \rangle$$

is a real-analytic function of s . Indeed, it is a finite linear combination of functions of the form $n^s m^{-s}$, for various non-zero integers n and m . As such, $v(s)$ extends to an entire function on the complex plane. If we replace \mathcal{H} by $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{H}$, then the operator δ^s can be defined for complex s . The complex analytic extension of v can then be defined by the same formula, if we extend the real inner product as a complex bilinear form on $\mathbb{C} \otimes \mathcal{H}$. For s on the imaginary axis, the operator δ^s is an isometry: its eigenvalues have norm 1. So (with x and y still fixed), we know that

$$|v(s)| \leq \|f\|_{\mathcal{C}^1} \|x\| \|y\|$$

for all s on the lines $\operatorname{Re}(s) = 1$ and $\operatorname{Re}(s) = -1$. On the strip $-1 \leq \operatorname{Re}(s) \leq 1$, the function $v(s)$ is bounded by some constant,

$$|v(s)| \leq K,$$

because $v(s)$ is a linear combination of real exponential functions n^s . To prove the Proposition, we need to show that we can take $K = \|f\|_{\mathcal{C}^1} \|x\| \|y\|$. This result follows from a lemma in complex analysis, called the ‘three lines lemma’, that we prove next. \square

Lemma 10. *If an entire function $v(s)$ is bounded in the strip $-1 \leq \operatorname{Re}(s) \leq 1$ and satisfies $|v(s)| \leq M$ on the two lines $\operatorname{Re}(s) = 1$ and $\operatorname{Re}(s) = -1$, then it satisfies $|v(s)| \leq M$ for all s in the strip.*

Proof. We are given that $v(s)$ is bounded in the strip, so suppose $|v(s)| \leq K$ for all s with $|\operatorname{Re} s| \leq 1$. For $\epsilon > 0$, consider the function

$$v_\epsilon(s) = e^{\epsilon s^2} v(s).$$

We have

$$|v_\epsilon(s)| \leq e^{\epsilon(1 - (\operatorname{Im} s)^2)} K$$

for all s in the strip, and

$$|v_\epsilon(s)| \leq e^\epsilon M$$

for all s on either of the two lines $\operatorname{Re}(s) = \pm 1$. In particular, if

$$t \geq \log(K/M)^{1/2}$$

then $|v_\epsilon(s)| \leq e^\epsilon M$ on all four edges of the rectangular region

$$|\operatorname{Re}(s)| \leq 1, |\operatorname{Im}(s)| \leq t.$$

By the maximum principle it now follows that $|v_\epsilon(s)| \leq e^\epsilon M$ for all s in the same rectangle. We can take t as large as we wish, so $|v_\epsilon(s)| \leq e^\epsilon M$ for all s in the strip. The lemma follows by letting ϵ go to zero. \square

Corollary 11. *For $-1 \leq s \leq 1$, the map $\mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ given by multiplication, $(f, h) \mapsto fh$ extends to a continuous bilinear map*

$$C^1(S^3; \mathbb{H}) \times \mathcal{H}_s \rightarrow \mathcal{H}_s.$$

\square

The other basic property of the Sobolev spaces \mathcal{H}_s that we need is the a version of the ‘‘Rellich Lemma’’. Note that \mathcal{H}_s is a subspace of \mathcal{H}_t if $s > t$.

Lemma 12. *Suppose $s > t$. If $\{u^{(n)}\}$ is a bounded sequence in \mathcal{H}_s , so that*

$$\|u^{(n)}\|_s \leq M$$

for all n , then there is a subsequence $\{u^{(n')}\}$ that is convergent in \mathcal{H}_t :

$$\|u^{(n')} - u\|_t \rightarrow 0$$

for some $u \in \mathcal{H}_t$.

(ii) A Fredholm operator

Recalling that E_{-1} and E_{-2} are zero, we define

$$\begin{aligned} \mathcal{H}^+ &= \hat{\bigoplus}_{d \geq 0} E_d \\ \mathcal{H}^- &= \hat{\bigoplus}_{d \leq -3} E_d, \end{aligned}$$

so that $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.

For $N \geq 1$, we also write $\mathcal{H}(N)$ for $L^2(S^3; \mathbb{H}^N)$. We have a decomposition

$$\mathcal{H}(N) = \mathcal{H}^+(N) \oplus \mathcal{H}^-(N).$$

Let $\Pi : \mathcal{H}(N) \rightarrow \mathcal{H}^+(N)$ be the projection. Because D is a homomorphism of left \mathbb{H} -modules, the summands are left \mathbb{H} -modules also. The same decomposition can be made for the space of polynomials,

$$\mathcal{E}(N) = \mathcal{E}^+(N) \oplus \mathcal{E}^-(N),$$

and we can form the completion in any Sobolev norm, to obtain

$$\mathcal{H}_s(N) = \mathcal{H}_s^+(N) \oplus \mathcal{H}_s^-(N)$$

as an orthogonal direct sum of Hilbert spaces. We continue to write Π for the projection to the first summand.

Let $GL(N; \mathbb{H})$ be the group of invertible N -by- N matrices with entries in \mathbb{H} . We can regard this as a subgroup of $GL(4N; \mathbb{R})$. The group $Sp(N)$ is the intersection $GL(N; \mathbb{H}) \cap O(4N)$. To each continuously differentiable map $f : S^3 \rightarrow Sp(N)$, we associate an operator on $\mathcal{H}_s^+(N)$ for $-1 \leq s \leq 1$ as follows:

Definition 13. Given a continuously differentiable function $f \in C^1(S^3; Sp(N))$, we define $A_f : \mathcal{H}_s^+(N) \rightarrow \mathcal{H}_s^+(N)$ for any s in the interval $-1 \leq s \leq 1$, by

$$A_f h = \Pi(fh).$$

Here, for each z in the sphere, $(fh)(z) = f(z)h(z)$ denotes the quaternion product of the matrix $f(z)$ and the quaternion column vector $h(z)$. By Corollary ??, this defines a continuous map

$$A : C^1(S^3; Sp(N)) \rightarrow \mathcal{A}_{\mathbb{R}}(\mathcal{H}_s^+(N)),$$

where $\mathcal{A}_{\mathbb{R}}(\mathcal{H}_s^+(N))$ denote the algebra of bounded, real-linear operators on the real Hilbert space $\mathcal{H}_s^+(N)$, equipped with the operator-norm topology.

Note that because we use left-multiplication of h by f in this definition, A_f is not a homomorphism of left \mathbb{H} -modules in general. However, in the special case that the entries of f are real, the \mathbb{H} structure is preserved, and we have a map to the algebra of bounded \mathbb{H} -linear maps:

$$A : C^1(S^3; O(N)) \rightarrow \mathcal{A}_{\mathbb{H}}(\mathcal{H}_s^+(N))$$

Recall that an operator is *Fredholm* if its kernel has finite dimension and its image has finite codimension. Let $\mathcal{F}_{\mathbb{R}}(\mathcal{H}_s^+(N))$ and $\mathcal{F}_{\mathbb{H}}(\mathcal{H}_s^+(N))$ denote the set of Fredholm maps in $\mathcal{A}_{\mathbb{R}}(\mathcal{H}_s^+(N))$ and $\mathcal{A}_{\mathbb{H}}(\mathcal{H}_s^+(N))$ respectively.

Theorem 14. *For every C^1 map $f : S^3 \rightarrow Sp(N)$, the operator A_f is a Fredholm operator on $\mathcal{H}_{1/2}^+(N)$. Thus A defines continuous maps*

$$\begin{aligned} A : C^1(S^3; Sp(N)) &\rightarrow \mathcal{F}_{\mathbb{R}}(\mathcal{H}_{1/2}^+(N)) \\ A : C^1(S^3; O(N)) &\rightarrow \mathcal{F}_{\mathbb{H}}(\mathcal{H}_{1/2}^+(N)). \end{aligned}$$

Proof.

□