Clifford modules

P. B. Kronheimer

For Math 272b, after Atiyah, Bott and Shapiro

1. Clifford modules

(i) Definitions

Let V be a finite-dimensional real inner product space. A (complex) *Clifford module* for V is a finite-dimensional complex inner product space S and a linear map

$$\gamma : V \to \mathrm{End}(S)$$

satisfying two conditions. The first is that $\gamma(v)$ is skew adjoint for all v:

$$\gamma(v)^* = -\gamma(v). \tag{1}$$

The second is the Clifford relation,

$$\gamma(u)\gamma(v) + \gamma(v)\gamma(u) = -2\langle u, v\rangle 1_S, \quad \forall u, v \in V. \tag{2}$$

If $e_1, \dots e_n$ is an orthonormal basis, the Clifford relation implies

$$\begin{aligned} \gamma(e_i)\gamma(e_i) &= -1 \\ \gamma(e_i)\gamma(e_j) &= -\gamma(e_j)\gamma(e_i) \end{aligned} \tag{3}$$

for $i \neq j$, and in fact the general Clifford relation follows from this second version, given that γ is linear.

Clifford modules (S, γ) and (S', γ') for V are *isomorphic* if there is a linear isometry $\phi : S \to S'$ such that $\phi\gamma(v) = \gamma'(v)\phi$ for all v. A linear subspace $T \subset S$ is *submodule* if it is invariant under $\gamma(v)$ for all v. A Clifford module S is *irreducible* if it has exactly two submodules: so S is non-zero and the only submodules are S and 0. Because $\gamma(v)$ is skew-adjoint, the orthogonal complement of a submodule is again a submodule. Hence:

Lemma 1.1. *Every Clifford module is a direct sum of irreducible Clifford modules.* □

We also have a version of "Shur's lemma" for Clifford modules:

Lemma 1.2. *Let S be an irreducible Clifford module. Then every isomorphisms $\phi : S \to S$ is of the form $\lambda 1_S$ for some scalar $\lambda \in S^1$. More generally every automorphisms ψ of $\mathbb{C}^r \otimes S$ has the form $\mu \otimes 1_S$ for some unitary r-by-r matrix μ.* □

(ii) Gradings

A complex inner product space S is $\mathbb{Z}/2$-*graded* if it is given an orthogonal direct sum decomposition

$$S = S^+ \oplus S^-. \tag{4}$$

We can think of the grading as being given by a unitary endomorphism

$$\epsilon : S \to S$$

with $\epsilon^2 = 1$, so that S^+ is the $+1$-eigenspace and S^- is the -1-eigenspace of ϵ. An endomorphism a of S is *even* if it commutes with ϵ (i.e. if a preserves the summands S^+ and S^-); we say a is *odd* if $a\epsilon = -\epsilon a$ (i.e. if $a(S^+) \subset S^-$ and $a(S^-) \subset S^+$).

We will say that a Clifford module S for V is a *graded Clifford module* if it is given a grading ϵ such that $\gamma(v)$ is odd for all v. If S is graded, we can write $\gamma(v)$ in block form with respect to the decomposition (4), as

$$\gamma(v) = \begin{bmatrix} 0 & -\sigma(v)^* \\ \sigma(v) & 0 \end{bmatrix}$$

where

$$\sigma(v) : S^+ \to S^-. \tag{5}$$

If the dimension n of V is even, then it turns out that any Clifford module has a grading. Pick an orthonormal basis $e_1, \dots e_n$ and define

$$\epsilon = (-i)^{(n/2)} \gamma(e_1) \dots \gamma(e_n). \tag{6}$$

Because $\gamma(e_i)$ commutes with itself and anticommutes with $\gamma(e_j)$ for $j \neq i$, and because n is even, this element ϵ anticommutes with $\gamma(e_i)$ for all i, as required. Also we check

$$\begin{aligned}
\epsilon^2 &= (-1)^{(n/2)} (\gamma(e_1) \dots \gamma(e_n))^2 \\
&= (-1)^{(n/2)} (-1)^{n(n-1)/2} \gamma(e_1)^2 \dots \gamma(e_n)^2 \\
&= (-1)^{(n/2)} (-1)^{n(n-1)/2} (-1)^n \\
&= 1.
\end{aligned}$$

Lemma 1.3. *Let S be an irreducible Clifford module for an even-dimensional vector space V. Then S has exactly two gradings, say ϵ and $-\epsilon$, one the negative of the other.*

Proof. Pick an orthonormal basis and define ϵ by the formula (6). Now let ϵ' be another grading. Because ϵ' commutes with $\gamma(e_i)$ and because ϵ is a product of an even number of such terms, it follows that ϵ and ϵ' commute. Their product $\epsilon\epsilon'$ therefore has square 1. Furthermore, $\epsilon\epsilon'$ commutes with $\gamma(v)$ for all v, so by Schur's Lemma, it follows that $\epsilon\epsilon'$ is a scalar with square 1. Thus $\epsilon' = \pm\epsilon$. □

Corollary 1.4. *If S is a Clifford module for an even-dimensional vector space V, and if ϵ is defined by the formula* (6)*, then ϵ depends only on the orientation class of the chosen orthonormal basis, not otherwise on the choice of e_i.*

Proof. For irreducible S, this follows from the previous lemma and the fact that interchanging two adjacent basis elements changes the sign of ϵ. For the general S, we apply Lemma 1.1. □

Definition 1.5. Let V be an even-dimensional, oriented vector space, and S a Clifford module. The *canonical grading* for S is defined by the operator ϵ given by (6), where $e_1, \ldots, e_n$ is any oriented orthonormal basis for V.

(iii) Classification

Here is the classification theorem for Clifford modules.

Theorem 1.6. *Let $V = \mathbb{R}^n$ be the standard real inner product space.*

(a) If $n = 2k$, then V has, up to isomorphism, a unique irreducible Clifford module (S_{2k}, γ_{2k}). Its dimension is 2^k.

(b) If $n = 2k + 1$, then V has, up to isomorphism, two irreducible Clifford modules, $(S_{2k+1}, \gamma_{2k+1})$ and $(S_{2k+1}, -\gamma_{2k+1})$, obtained from each other by changing the sign of γ. The dimension of S_{2k+1} is 2^k.

Remark. In the case that V has even dimension, (S, γ) and $(S, -\gamma)$ are always isomorphic. This is because $\epsilon : S \to S$ anticommutes with γ, so is an isomorphism from (S, γ) to $(S, -\gamma)$. In the odd case, one can see that (S, γ) and $(S, -\gamma)$ are not isomorphic when S is irreducible by looking at the product $\gamma(e_1) \ldots \gamma(e_n)$. This commutes with all $\gamma(e_i)$ when n is odd, so by Schur's Lemma it is a scalar. Changing the sign of all the $\gamma(e_i)$ changes the sign of this scalar.

Proof. We start by verifying the theorem bare-hands for $n = 1$ and $n = 2$. For $n = 1$, we take $S_1 = \mathbb{C}$, and we set $\gamma_1(e_1) = i$. Since $\gamma_1(e_1)^2$ needs to be -1, our choice is one of only two possibilities. For a general Clifford module (S, γ) for $\mathbb{R}$, we can diagonalize $\gamma(e_1)$: in some orthonormal basis, it has diagonal entries $\pm i$, because its square is -1. This expresses S as a direct some of copies of the two irreducible representations.

When $n = 2$, there can be no 1-dimensional Clifford module, because $\gamma(e_1)$ and $\gamma(e_2)$ need to anticommute. So we seek a 2-dimensional module, decomposed using the canonical ϵ as $S^+ \oplus S^- = \mathbb{C} \oplus \mathbb{C}$. Consider $\sigma(e_1)$ and $\sigma(e_2)$, as in (5). We can choose the bases so that the unitary transformation $\sigma(e_1)$ is 1. We want

$$(-i)\gamma(e_1)\gamma(e_2) = \epsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and this forces $\sigma(e_2) = -i$. So there is indeed a unique 2-dimensional Clifford module (S_2, γ_2) up to isomorphism: it is given by

$$\gamma_2(e_1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \gamma_2(e_2) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.$$

If (S, γ) is a general Clifford module for $\mathbb{R}^2$, we again decompose it as $S^+ \oplus S^-$. Then $\sigma(e_1)$ is an isomorphism between the two summands, so we can write $S^+ = S^- = \mathbb{C}^r$ and $\gamma(e_1)$ has the block form

$$\gamma(e_1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The fact that $-i\gamma(e_1)\gamma(e_2)$ is ϵ again forces

$$\gamma_2(e_2) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.$$

Thus the Clifford module S is isomorphic to $\mathbb{C}^r \otimes S_2$: that is, r copies of the standard module.

We now construct a Clifford module (S_n, γ_n) of the correct dimension for $\mathbb{R}^n$, for all n. The tool is a general tensor product construction. Suppose $V = V_1 \oplus V_2$, and let (S_1, γ_1), (S_2, γ_2) be Clifford modules. Suppose V_1 is even-dimensional and oriented, and let ϵ be the canonical grading operator for S_1. Then we can construct a Clifford module (S, γ) for V by setting

$$\begin{aligned} S &= S_1 \otimes S_2 \\ \gamma &= \gamma_1 \otimes 1 + \epsilon \otimes \gamma_2. \end{aligned}$$

We must verify the Clifford relation: so let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be elements of V. We compute

$$\begin{aligned} \gamma(u)\gamma(v) + \gamma(v)\gamma(u) &= \big(\gamma_1(u_1) \otimes 1 + \epsilon \otimes \gamma_2(u_2)\big)\big(\gamma_1(v_1) \otimes 1 + \epsilon \otimes \gamma_2(v_2)\big) \\ &\quad + \big(\gamma_1(v_1) \otimes 1 + \epsilon \otimes \gamma_2(v_2)\big)\big(\gamma_1(u_1) \otimes 1 + \epsilon \otimes \gamma_2(u_2)\big) \\ &= (\gamma_1(u_1)\gamma_1(v_1) + \gamma_1(v_1)\gamma_1(u_1)) \otimes 1 \\ &\quad + \epsilon^2 \otimes (\gamma_2(u_2)\gamma_2(v_2) + \gamma_2(v_2)\gamma_2(u_2)) \\ &\quad + \gamma_1(u_1)\epsilon \otimes \gamma_2(v_2) + \epsilon\gamma_1(u_1) \otimes \gamma_2(v_2) \\ &\quad + \epsilon\gamma_1(v_1) \otimes \gamma_2(u_2) + \gamma_1(v_1)\epsilon \otimes \gamma_2(u_2) \\ &= (\gamma_1(u_1)\gamma_1(v_1) + \gamma_1(v_1)\gamma_1(u_1)) \otimes 1 \\ &\quad + 1 \otimes (\gamma_2(u_2)\gamma_2(v_2) + \gamma_2(v_2)\gamma_2(u_2)) \\ &= -2\big(\langle u_1, v_1\rangle + \langle u_2, v_2\rangle\big) 1 \otimes 1 \\ &= -2\langle u, v\rangle 1_S, \end{aligned}$$

which is what we needed.

Using this tensor product construction, we define (S_n, γ_n) inductively for $n \geq 3$, in terms of the cases $n = 1$ and $n = 2$ which we defined above: we set

$$\begin{aligned} S_n &= S_2 \otimes S_{n-2} \\ \gamma_n &= \gamma_2 \otimes 1 + \epsilon_2 \otimes \gamma_{n-2} \end{aligned}$$

using the decomposition $\mathbb{R}^n = \mathbb{R}^2 \oplus \mathbb{R}^{n-2}$. Here S_2 is the standard Clifford module for $\mathbb{R}^2$ described above, and $\epsilon_2 = -i\gamma_2(e_1)\gamma_2(e_2)$ is its canonical grading.

If (S,γ) is a Clifford module, let (S',γ') denote the same space with the sign of γ reversed. Let (S,γ) be a Clifford module for $\mathbb{R}^n$. We wish to show it is a sum of copies of the standard S_n and its companion S'_n (which may be isomorphic, if n is even). The proof is by induction on n for $n \geq 3$. Write $\mathbb{R}^n = \mathbb{R}^2 \oplus \mathbb{R}^{n-2}$, and set $\delta = -i\gamma(e_1)\gamma(e_2)$.

This element δ has square 1, and commutes with $\gamma(v)$ for v in $\mathbb{R}^{n-2}$. Also, δ anticommutes with $\gamma(e_1)$. It follows that if we decompose S into the $+1$ and -1 eigenspaces of δ, as

$$S = S_a \oplus S_b,$$

then each of S_a and S_b is a Clifford module for $\mathbb{R}^{n-2}$. Furthermore, $\gamma(e_1)$ provides an isomorphism of vector spaces $\phi : S_a \to S_b$. Because $\gamma(e_1)$ anticommutes with $\gamma(v)$ for $v \in \mathbb{R}^{n-2}$, this ϕ is an isomorphism between S_a and S'_b as Clifford modules for $\mathbb{R}^{n-2}$.

We have seen that there is a Clifford module S_a for $\mathbb{R}^{n-2}$ such that $S = S_a \oplus S'_a$ as Clifford modules for $\mathbb{R}^{n-2}$ and such that

$$\gamma(e_1) = \begin{bmatrix} 0 & -1 \\ 1 & 0. \end{bmatrix}$$

The relation $\delta = -i\gamma(e_1)\gamma(e_2)$ once again determines $\gamma(e_2)$. So S is entirely determined by S_a. In particular, S is irreducible if and only if S_a is; and our analysis of the cases $n = 1$ and 2 implies that there is exactly one irreducible module if n is even and two if n is odd. □

2. Real and quaternionic structures

(i) Real Clifford modules

Let V again be real inner product space. A *J-Clifford module* for V is a Clifford module (S,γ) with a conjugate-linear automorphism J. That is, J is a linear map

$$J : S \to \bar{S}$$

commuting with $\gamma(v)$ for all v. For J-Clifford modules (S,γ,J), we have the evident notions of isomorphism, submodules and irreducibility.

Suppose (S,γ) is an irreducible complex Clifford module, and has such a J. The operator $J^2 : S \to S$ is complex-linear, and is an automorphism of S. It follows that J^2 is a scalar in S^1, by Schur's Lemma. Because J^2 commutes with J, this scalar is real. Thus

$$J^2 = \pm 1.$$

If $J^2 = 1$, we call the structure *real*; if $J^2 = -1$, we call the structure quaternionic. If J' and J are two conjugate linear automorphisms of (S,γ), then Schur's Lemma similarly implies that $J' = \lambda J$, for some λ in the circle. Multiplication by $\lambda^{1/2}$ is then an isomorphism

$$(S,\gamma,J) \to (S,\gamma,J').$$

(ii) Classifying J-Clifford modules

We will now classify J-Clifford modules in the case $n = \dim V$ is even. We shall prove:

Proposition 2.1. *If n is even, the standard Clifford module (S_n, γ_n) for $\mathbb{R}^n$ admits a conjugate-linear automorphism $J_n : (S_n, \gamma_n) \to (\bar{S}_n, \gamma_n)$.*

In light of the above remarks, we have:

Corollary 2.2. *Up to isomorphism, $\mathbb{R}^n$ has a unique, irreducible J-Clifford module (S^n, γ_n, J_n) if n is even.* □

Proof of Proposition 2.1. Before constructing J_n, we make the following observation. As well as the real versus quaternionic dichotomy, we can ask in the case that the dimension of V is even whether J commutes with the canonical grading operator ϵ. For this quesion, we have the following result:

Lemma 2.3. *Let V have real dimension $n = 2k$ and a chosen orientation. Let (S, γ) be a Clifford module for V with a conjugate-linear automorpism J. Then J commutes with the canonical grading operator ϵ if and only if k is even: indeed,*

$$\epsilon J = (-1)^k J\epsilon.$$

Proof. This follows at once from the formula (6) for ϵ, and the fact that J commutes with $\gamma(v)$ and anticommutes with i. □

Returning to the proof of the Proposition, we now construct J_n for all even n, inductively. For $n = 2$, we recall that $S_2 = \mathbb{C} \oplus \mathbb{C}$, and we set

$$J_2(a, b) = (-\bar{b}, \bar{a}).$$

It is straightforward to verify that this commutes with $\gamma_2(e_i)$ for $i = 1, 2$.

For the induction step, suppose we have $J_n : S_n \to \bar{S}_n$ inductively. Write $n = 2k$. We recall that $S_{n+2} = S_2 \otimes S_n$, and we set

$$J_{n+2} = \begin{cases} J_2 \otimes \epsilon_n J_n, & \text{if } k = n/2 \text{ is odd} \\ J_2 \otimes J_n, & \text{if } k = n/2 \text{ is even.} \end{cases}$$

We must check that J_{n+2} commutes with $\gamma_{n+2}(v)$. write $v = (z, w)$ in $\mathbb{R}^2 \oplus \mathbb{R}^n$. In the case k odd, we have

$$\begin{aligned} J_{n+2}\gamma_{n+2}(v) &= (J_2 \otimes \epsilon_n J_n)(\gamma_2(z) \otimes 1 + \epsilon_2 \otimes \gamma_n(w)) \\ &= J_2\gamma_2(z) \otimes \epsilon_n J_n + J_2\epsilon_2 \otimes \epsilon_n J_n \gamma_n(w) \\ &= \gamma_2(z) J_2 \otimes \epsilon_n J_n + (-\epsilon_2 J_2) \otimes (-J_n \epsilon_n) \\ &= \gamma_2(z) J_2 \otimes \epsilon_n J_n + \epsilon_2 J_2 \otimes J_n \epsilon_n \\ &= (\gamma_2(z) \otimes 1 + \epsilon_2 \otimes \gamma_n(w))(J_2 \otimes \epsilon_n J_n) \\ &= \gamma_{n+2}(v) J_{n+2}. \end{aligned}$$

In the third line, we used the lemma above, to tell us that J_n anticommutes with ϵ_n in this case, when k is odd. The verification in the case that k is even is similar. □

(iii) The eight-step periodicity

We now further examine the irreducible J-Clifford module (S_n, γ_n, J_n) for $\mathbb{R}^n$ for even n. With $n = 2k$, we have seen that J_n is even or odd (that is, it commutes or anticommutes with the grading operator ϵ_n) according as k is even or odd respectively. We have not yet examined whether J_n is real or quaternionic.

Suppose first that $n = 2k$ with k *even*, so that

$$J_{n+2} = J_2 \otimes J_n.$$

Then

$$\begin{aligned} J_{n+2}^2 &= J_2^2 \otimes J_n^2 \\ &= (-1) \otimes J_n^2. \end{aligned}$$

Thus in the case k even, J_{n+2} is of quaternion type if J_n is real, and vice versa. When k is odd on the other hand,

$$\begin{aligned} J_{n+2}^2 &= J_2^2 \otimes (\epsilon_n J_n)^2 \\ &= (-1) \otimes (\epsilon_n J_n \epsilon_n J_n) \\ &= (-1) \otimes (-\epsilon_n \epsilon_n J_n J_n) \\ &= 1 \otimes J_n^2. \end{aligned}$$

So when k is odd, J_{n+2} and J_n have the same type.

The following table summarizes how the type of J_{n+2} depends on the type of J_n. We write $\mathbb{R}$ or $\mathbb{H}$ to indicate whether J_n is real or quaternionic, and we write 'even' or 'odd' to indicate whether J_n commutes or anticommutes with ϵ_n.

J_n	even	even	odd	odd
	$\mathbb{R}$	$\mathbb{H}$	$\mathbb{R}$	$\mathbb{H}$
J_{n+2}	odd	odd	even	even
	$\mathbb{H}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{H}$

Starting with J_2 that is odd an quaternionic, we tabulate how things develop:

n		
2	odd	$\mathbb{H}$
4	even	$\mathbb{H}$
6	odd	$\mathbb{R}$
8	even	$\mathbb{R}$

Thereafter, the pattern repeats: the type only depends on n modulo 8.

(iv) Real and quaternionic structures in the odd-dimensional case

If the dimension, n, of the inner product space V is odd, then there are two, non-isomorphic, irreducible complex Clifford modules, differing in the sign of γ. (See Theorem 1.6.) Let us call these (S, γ) and (S', γ'). We can distinguish between these two by looking at an automorphism η defined by a formula like the formula (6) defining ϵ: if $n = 2k - 1$, we set

$$\eta = (-i)^k \gamma(e_1)\gamma(e_2)\ldots\gamma(e_n).$$

This η commutes with $\gamma(e_i)$ for all i; and $\eta^2 = 1$. It follows that $\eta = \pm 1$ on the irreducible Clifford module S. We distinguish S from S' by declaring S to be such that $\eta = +1$.

Much as in the case that n is even, one establishes inductively the following facts about conjugate-linear automorphisms J.

(a) When n is 3 or 7 mod 8, the irreducible complex Clifford module (S, γ) for $\mathbb{R}^n$ has a unique conjugate-linear automorphism J. The automorphism J is quaternionic in the case $n \equiv 3$ mod 8 and is real in the case $n \equiv 7$ mod 8.

(b) When n is 1 or 5 mod 8, there is an isomorphism of Clifford modules $J : S \to \bar{S}'$. Thus the Clifford module $S \oplus S'$ carries a real structure given by the conjugate-linear automorphism

$$\begin{bmatrix} 0 & J^{-1} \\ J & 0 \end{bmatrix}$$

(which has square 1). There is equally a quaternionic structure given by the conjugate-linear automorphism

$$\begin{bmatrix} 0 & -J^{-1} \\ J & 0 \end{bmatrix},$$

which has square -1. We take the first of these to be the *standard* J-Clifford module in this dimension. It is the unique, irreducible real J-Clifford module for $\mathbb{R}^n$.

3. The spin groups

Fix n not equal to 1 or 5 mod 8. Let (S, γ, J) be the standard irreducible J-Clifford module for $\mathbb{R}^n$. In the case that n is even, this is the unique irreducible J-Clifford module. In the case that n is 3 or 7 mod 8, we have to use η to distinguish the two possible irreducible J-Clifford modules by the sign of γ. In both cases, S is irreducible as a complex Clifford module: its only automorphisms as a complex Clifford module are the unit scalars S^1; and as a J-Clifford module its only automorphisms are ± 1.

Let $U(S)$ denote the group of unitary transformations of S. Define

$$\text{Spin}(n) \subset SO(n) \times U(S)$$

to be the pairs (a, A) such that:

(a) $A : S \to S$ commutes with J;

(b) the diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\gamma} & \mathrm{End}(S) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle B \mapsto ABA^{-1}} \\ \mathbb{R}^n & \xrightarrow{\gamma} & \mathrm{End}(S) \end{array}$$

commutes

These conditions define a subgroup of the product, and there is a group homomorphism $\pi : \mathrm{Spin}(n) \to SO(n)$ defined by $(a, A) \mapsto a$.

Lemma 3.1. *The map* $\pi : \mathrm{Spin}(n) \to SO(n)$ *is surjective and has kernel the* 2*-element group* $\{(1, 1), (1, -1)\}$.

Proof. If (a, A) is in the kernel, then a is 1 and A is an automorphism of S preserving γ and J. So A is ± 1. So the kernel is as described. If a belongs to $SO(n)$, consider the irreducible J-Clifford module $(S, \tilde{\gamma}, J)$, where $\tilde{\gamma}(v) = \gamma(av)$. By uniqueness of the irreducible J-Clifford module, there is an isomorphism of J-Clifford modules

$$A : (S, \gamma, J) \to (S, \tilde{\gamma}, J).$$

Then (a, A) belongs to $\mathrm{Spin}(n)$ and $\pi(a, A) = a$. So π is surjective. □

Lemma 3.2. *The map* $\pi : \mathrm{Spin}(n) \to SO(n)$ *is the non-trivial double cover of* $SO(n)$.

Proof. We verify this in the basic case $n = 2$. We write an element (x, y) in $\mathbb{R}^2$ in complex coordinates, as $z = x + iy$. Then the Clifford module for $\mathbb{R}^2$ has the form

$$\gamma(z) = \begin{bmatrix} 0 & -z \\ \bar{z} & 0. \end{bmatrix}$$

We can identify an element a in $SO(2)$ as a unit complex number acting by $z \mapsto az$. To find elements (a, A) of $\mathrm{Spin}(2)$ we seek $A : \mathbb{C}^2 \to \mathbb{C}^2$ commuting with J_2 and satisfying

$$A \begin{bmatrix} 0 & -z \\ \bar{z} & 0 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & -az \\ \bar{a}\bar{z} & 0 \end{bmatrix}.$$

There are two solutions, corresponding to the two complex square roots of a:

$$A = \begin{bmatrix} a^{1/2} & 0 \\ 0 & \bar{a}^{1/2} \end{bmatrix}.$$

Thus $\mathrm{Spin}(2)$ is the subgroup of $S^1 \times U(2)$ given by the elements

$$\left(b^2, \begin{bmatrix} b & 0 \\ 0 & \bar{b} \end{bmatrix} \right), \qquad b \in S^1.$$

This is the non-trivial double-cover of the circle. □