

Math 281 Lecture I

Semi-classical analysis.

This course represents joint work in progress with Victor Guillemin. An earlier version can be found in Chapters I and VII of our book “Geometric Asymptotic” which is available on the web for free downloading.

The phrase “semi-classical” will have the following meaning: In the solution of certain partial differential equations - especially hyperbolic equations - the method of bicharacteristics starts off with a system of ordinary differential equations. For example, the bicharacteristics for the equations of wave optics are the differential equations of geometrical optics. These can be regarded as the “zeroth” order or “classical approximation”. Similarly, the classical approximation to the Schrodinger equation of quantum mechanics give the ordinary differential equations of classical mechanics.

In terms of an approximation scheme that I hope to describe in today’s lecture, there is then a “first order”, “second order” etc. series of approximation steps. These are known as the “semi-classical” approximation. The thrust of this course will be to study this “semi-classical world” in its own right. We will find that it has some wonderful properties and geometric consequences.

Hyperbolic operators.

Let $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with coordinates (x^1, \dots, x^n, t) . Let

$$P = P \left(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right)$$

be a k -th order partial differential operator. Suppose that we want to solve the partial differential equation

$$Pu = 0$$

with initial conditions

$$u(x, 0) = \delta_0, \quad \frac{\partial^i}{\partial t^i} u(x, 0) = 0, \quad i = 1, \dots, k - 1.$$

Let ρ be a C^∞ function of x of compact support which is identically one near the origin. We can write

$$\delta_0(x) = \frac{1}{(2\pi)^n} \rho(x) \int_{\mathbb{R}^n} e^{ix \cdot \xi} d\xi.$$

Let us introduce polar coordinates in ξ space:

$$\xi = \omega \cdot r, \quad \|\omega\| = 1, \quad r = \|\xi\|$$

so we can rewrite the above expression as

$$\delta_0(x) = \frac{1}{(2\pi)^n} \rho(x) \int_{\mathbb{R}_+} \int_{S^{n-1}} e^{i(x \cdot \omega)r} r^{n-1} dr d\omega$$

where $d\omega$ is the measure on the unit sphere S^{n-1} . This shows that we are interested in solving the partial differential equation $Pu = 0$ with the initial conditions

$$u(x, 0) = \rho(x) e^{i(x \cdot \omega)r} r^{n-1}, \quad \frac{\partial^i}{\partial t^i} u(x, 0) = 0, \quad i = 1, \dots, k-1.$$

The problem.

More generally, set

$$r = \hbar^{-1}$$

and let

$$\psi \in C^\infty(\mathbb{R}^n).$$

We look for solutions of the partial differential equation with initial conditions

$$Pu(x, t) = 0, \quad u(x, 0) = \rho(x)e^{i\frac{\psi(x)}{\hbar}} \hbar^{-\ell} \quad \frac{\partial^i}{\partial t^i} u(x, 0) = 0, \quad i = 1, \dots, k - 1. \quad (1.1)$$

The eikonal equation.

We look for solutions of the partial differential equation with initial conditions

$$Pu(x, t) = 0, \quad u(x, 0) = \rho(x)e^{i\frac{\psi(x)}{\hbar}} \hbar^{-\ell} \quad \frac{\partial^i}{\partial t^i} u(x, 0) = 0, \quad i = 1, \dots, k - 1. \quad (1.1)$$

Look for solutions of (1.1) of the form

$$u(x, t) = a(x, t, \hbar)e^{i\phi(x, t)/\hbar} \quad (1.2)$$

where

$$a(x, t, \hbar) = \hbar^{-\ell} \sum_{i=0}^{\infty} a_i(x, t) \hbar^i. \quad (1.3)$$

The principal symbol.

Define the **principal symbol** $H(x, t, \xi, \tau)$ of the differential operator P by

$$\hbar^k e^{-i\frac{x \cdot \xi + t\tau}{\hbar}} P e^{i\frac{x \cdot \xi + t\tau}{\hbar}} = H(x, t, \xi, \tau) + O(\hbar). \quad (1.4)$$

We think of H as a function on $T^*\mathbb{R}^{n+1}$.

If we apply P to $u(x, t) = a(x, t, \hbar) e^{i\phi(x, t)/\hbar}$, then the term of degree \hbar^{-k} is obtained by applying all the differentiations to $e^{i\phi(x, t)/\hbar}$. In other words,

$$\hbar^k e^{-i\phi/\hbar} P a(x, t) e^{i\phi/\hbar} = H \left(x, t, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial t} \right) a(x, t) + O(\hbar). \quad (1.5)$$

So as a first step we must solve the first order non-linear partial differential equation

$$H \left(x, t, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial t} \right) = 0 \quad (1.6)$$

for ϕ . Equation (1.6) is known as the **eikonal equation** and a solution ϕ to (1.6) is called an **eikonal**. The Greek word eikona $\epsilon\iota\kappa\omega\nu\alpha$ means image.

Hyperbolicity.

For all (x, t, ξ) the function

$$\tau \mapsto H(x, t, \xi, \tau)$$

is a polynomial of degree (at most) k in τ . We say that P is **hyperbolic** if this polynomial has k distinct real roots

$$\tau_i = \tau_i(x, t, \xi).$$

These are then smooth functions of (x, t, ξ) .

We assume from now on that P is hyperbolic.

Initial manifolds.

$$H \left(x, t, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial t} \right) = 0 \quad (1.6)$$

We assume from now on that P is hyperbolic. For each $i = 1, \dots, k$ let

$$\Sigma_i \subset T^*\mathbb{R}^{n+1}$$

be defined by

$$\Sigma_i = \{(x, 0, \xi, \tau) \mid \xi = d_x \psi, \tau = \tau_i(x, 0, \xi)\} \quad (1.7)$$

where ψ is the function occurring in the initial conditions in (1.1). The classical method for solving (1.6) is to reduce it to solving a system of ordinary differential equations with initial conditions given by (1.7). We recall the method:

The canonical one form on the cotangent bundle.

If X is a differentiable manifold, then its cotangent bundle T^*X carries a **canonical one form** $\alpha = \alpha_X$ defined as follows: Let

$$\pi : T^*X \rightarrow X$$

be the projection sending any covector $p \in T_x^*X$ to its base point x . If $v \in T_p(T^*X)$ is a tangent vector to T^*X at p , then

$$d\pi_p v$$

is a tangent vector to X at x . In other words, $d\pi_p v \in T_x X$. But $p \in T_x^*X$ is a linear function on $T_x X$, and so we can evaluate p on $d\pi_p v$. The canonical linear differential form α is defined by

$$\langle \alpha, v \rangle := \langle p, d\pi_p v \rangle \quad \text{if } v \in T_p(T^*X). \quad (1.8)$$

For example, if our manifold is \mathbb{R}^{n+1} as above, so that we have coordinates (x, t, ξ, τ) on $T^*\mathbb{R}^{n+1}$ the canonical one form is given in these coordinates by

$$\alpha = \xi \cdot dx + \tau dt = \xi_1 dx^1 + \cdots + \xi_n dx^n + \tau dt. \quad (1.9)$$

The canonical two form on the cotangent bundle.

This is defined as

$$\omega_X = -d\alpha_X. \quad (1.10)$$

Let q^1, \dots, q^n be local coordinates on X . Then dq^1, \dots, dq^n are differential forms which give a basis of T_x^*X at each x in the coordinate neighborhood U . In other words, the most general element of T_x^*X can be written as $p_1(dq^1)_x + \dots + p_n(dq^n)_x$. Thus $q^1, \dots, q^n, p_1, \dots, p_n$ are local coordinates on

$$\pi^{-1}U \subset T^*X.$$

In terms of these coordinates the canonical one form is given by

$$\alpha = p \cdot dq = p_1 dq^1 + \dots + p_n dq^n$$

Hence the canonical two form has the local expression

$$\omega = dq \wedge \cdot dp = dq^1 \wedge dp_1 + \dots + dq^n \wedge dp_n. \quad (1.11)$$

The form ω is closed and is of maximal rank, i.e. ω defines an isomorphism between the tangent space and the cotangent space at every point of T^*X .

Symplectic manifolds.

A two form which is closed and is of maximal rank is called **symplectic**. A manifold M equipped with a symplectic form is called a **symplectic manifold**. We shall study some of the basic geometry of symplectic manifolds in Chapter 2. But here are some elementary notions which follow directly from the definitions: A diffeomorphism $f : M \rightarrow M$ is called a **symplectomorphism** if $f^*\omega = \omega$. More generally if (M, ω) and (M', ω') are symplectic manifolds then a diffeomorphism

$$f : M \rightarrow M'$$

is called a symplectomorphism if

$$f^*\omega' = \omega.$$

Symplectic vector fields.

If v is a vector field on M , then the general formula for the Lie derivative of a differential form Ω with respect to v is given by

$$D_v\Omega = i(v)d\Omega + di(v)\Omega.$$

This is known as Weil's identity. See (??) in Chapter ?? below. If we take Ω to be a symplectic form ω , so that $d\omega = 0$, this becomes

$$D_v\omega = di(v)\omega.$$

So the flow $t \mapsto \exp tv$ generated by v consists of symplectomorphisms if and only if

$$di(v)\omega = 0.$$

Hamiltonian vector fields.

In particular, if H is a function on a symplectic manifold M , then the **Hamiltonian vector field** v_H associated to H and defined by

$$i(v_H)\omega = -dH \quad (1.12)$$

satisfies

$$(\exp tv_H)^*\omega = \omega.$$

Also

$$D_{v_H}H = i(v_H)dH = -i(v_H)i(v_H)\omega = \omega(v_H, v_H) = 0.$$

Thus

$$(\exp tv_H)^*H = H. \quad (1.13)$$

So the flow $\exp tv_H$ preserves the level sets of H . In particular, it carries the zero level set - the set $H = 0$ - into itself.

Isotropic submanifolds.

A submanifold Y of a symplectic manifold is called **isotropic** if the restriction of the symplectic form ω to Y is zero. So if

$$\iota_Y : Y \rightarrow M$$

denotes the injection of Y as a submanifold of M , then the condition for Y to be isotropic is

$$\iota_Y^* \omega = 0$$

where ω is the symplectic form of M .

For example, consider the submanifold Σ_i of $T^*(\mathbb{R}^{n+1})$ defined by (1.7). According to (1.9), the restriction of $\alpha_{\mathbb{R}^{n+1}}$ to Σ_i is given by

$$\frac{\partial \psi}{\partial x_1} dx_1 + \cdots + \frac{\partial \psi}{\partial x_n} dx_n = d_x \psi$$

since $t \equiv 0$ on Σ_i . So

$$\iota_{\Sigma_i}^* \omega_{\mathbb{R}^{n+1}} = -d_x d_x \psi = 0$$

and hence Σ_i is isotropic.

The flow-out of an isotropic submanifold.

Let H be a smooth function on a symplectic manifold M and let Y be an isotropic submanifold of M contained in a level set of H . For example, suppose that

$$H|_Y \equiv 0. \quad (1.14)$$

Consider the submanifold of M swept out by Y under the flow $\exp tv_\xi$. More precisely suppose that

- v_H is transverse to Y in the sense that for every $y \in Y$, the tangent vector $v_H(y)$ does *not* belong to $T_y Y$ and
- there exists an open interval I about 0 in \mathbb{R} such that $\exp tv_H(y)$ is defined for all $t \in I$ and $y \in Y$.

We then get a map

$$j : Y \times I \rightarrow M, \quad j(y, t) := \exp tv_H(y)$$

which allows us to realize $Y \times I$ as a submanifold Z of M . The tangent space to Z at a point (y, t) is spanned by

$$(\exp tv_H)_*TY_y \quad \text{and} \quad v_H(\exp tv_H y)$$

and so the dimension of Z is $\dim Y + 1$.

Proposition 1 *With the above notation and hypotheses, Z is an isotropic submanifold of M .*

Proposition 1 *With the above notation and hypotheses, Z is an isotropic submanifold of M .*

Proof. We need to check that the form ω vanishes when evaluated on

1. two vectors belonging to $(\exp tv_H)_*TY_y$ and
2. $v_H(\exp tv_H y)$ and a vector belonging to $(\exp tv_H)_*TY_y$.

For the first case observe that if $w_1, w_2 \in T_y Y$ then

$$\omega((\exp tv_H)_*w_1, (\exp tv_H)_*w_2) = (\exp tv_H)^*\omega(w_1, w_2) = 0$$

since

$$(\exp tv_H)^*\omega = \omega$$

and Y is isotropic.

For the second case observe that $i(v_H)\omega = -dH$ and so for $w \in T_y Y$ we have

$$\omega((\exp tv_H)_*w, i(v_H(\exp tv_H y))) = dH(w) = 0$$

since H is constant on Y . \square

Application to our problem.

If we consider the function H arising as the symbol of a hyperbolic equation, i.e. the function H given by (1.4) then H is a homogeneous polynomial in ξ and τ of the form $b(x, t, \xi) \prod_i (\tau - \tau_i)$, with $b \neq 0$ so

$$\frac{\partial H}{\partial \tau} \neq 0 \quad \text{along} \quad \Sigma_i.$$

But the coefficient of $\partial/\partial t$ in v_H is $-\partial H/\partial \tau$. Now $t \equiv 0$ along Σ_i so v_H is transverse to Σ_i . Our transversality condition is satisfied. We can arrange that the second of our conditions, the existence of solutions for an interval I can be satisfied locally. (In fact, suitable compactness conditions that are frequently satisfied will guarantee the existence of global solutions.)

Thus, at least locally, the submanifold of $T^*(\mathbb{R}^{n+1})$ swept out from Σ_i by $\exp tv_H$ is an $n + 1$ dimensional isotropic submanifold.

Lagrangian submanifolds.

A submanifold of a symplectic manifold which is isotropic and whose dimension is one half the dimension of M is called **Lagrangian**. We shall study Lagrangian submanifolds in detail in Chapter 2. Here we shall show how they are related to our problem of solving the eikonal equation (1.6).

The submanifold Σ_i of $T^*\mathbb{R}^{n+1}$ is isotropic and of dimension n . It is transversal to v_H . Therefore the submanifold Λ_i swept out by Σ_i under $\exp tv_H$ is Lagrangian. Also, near $t = 0$ the projection

$$\pi : T^*\mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$$

when restricted to Λ_i is (locally) a diffeomorphism. It is (locally) **horizontal** in the sense of the next section.

Lagrangian submanifolds of the cotangent bundle.

To say that a submanifold $\Lambda \subset T^*X$ is Lagrangian means that Λ has the same dimension as X and that the restriction to Λ of the canonical one form α_X is closed.

Suppose that Z is a submanifold of T^*X and that the restriction of $\pi : T^*X \rightarrow X$ to Z is a diffeomorphism. This means that Z is the image of a section

$$s : X \rightarrow T^*X.$$

Such a section is the same as assigning a covector at each point of X , in other words it is a linear differential form. For the purposes of the discussion we temporarily introduce a redundant notation and call the section s by the name β_s when we want to think of it as a linear differential form. We claim that

$$s^*\alpha_X = \beta_s.$$

Indeed, if $w \in T_xX$ then $d\pi_{s(x)} \circ ds_x(w) = w$ and hence

$$s^*\alpha_X(w) = \langle (\alpha_X)_{s(x)}, ds_x(w) \rangle = \langle s(x), d\pi_{s(x)}ds_x(w) \rangle = \langle s(x), w \rangle = \beta_s(x)(w).$$

Thus the submanifold Z is Lagrangian if and only if $d\beta_s = 0$.

Lagrangian submanifolds of the cotangent bundle, 2.

Thus the submanifold Z is Lagrangian if and only if $d\beta_s = 0$. Let us suppose that X is connected and simply connected. Then $d\beta = 0$ implies that $\beta = d\phi$ where ϕ is determined up to an additive constant.

With some slight abuse of language, let us call a Lagrangian submanifold of T^*X **horizontal** if the restriction of $\pi : T^*X \rightarrow X$ to Λ is a diffeomorphism. We have proved

Proposition 2 *Suppose that X is connected and simply connected. Then every horizontal Lagrangian submanifold of T^*X is given by a section $\gamma_\phi : X \rightarrow T^*X$ where γ_ϕ is of the form*

$$\gamma_\phi(x) = d\phi(x)$$

where ϕ is a smooth function determined up to an additive constant.

Local solution of the eikonal equation.

We have now found a local solution of the eikonal equation! Starting with the initial conditions Σ_i given by (1.7) at $t = 0$, we obtain the Lagrangian submanifold Λ_i . Locally (in x and in t near zero) the manifold Λ_i is given as the image of γ_{ϕ_i} for some function ϕ_i . The fact that Λ_i is contained in the set $H = 0$ then implies that ϕ_i is a solution of (1.6).

Caustics!

What can go wrong globally? One problem that might arise is with integrating the vector field v_H . As is well known, the existence theorem for non-linear ordinary differential equations is only local - solutions might “blow up” in a finite interval of time. In many applications this is not a problem because of compactness or boundedness conditions. A more serious problem - one which will be a major concern of this book - is the possibility that after some time the Lagrangian manifold is no longer horizontal.

If $\Lambda \subset T^*X$ is a Lagrangian submanifold, we say that a point $m \in \Lambda$ is a **caustic** if

$$d\pi_m T_m \Lambda \rightarrow T_x X. \quad x = \pi(m)$$

is *not* surjective. A key ingredient in what we will need to do is to describe how to choose convenient parametrizations of a Lagrangian manifolds near caustics. The first person to deal with this problem (through the introduction of so-called “angle characteristics”) was Hamilton (1805-1865) in a paper he communicated to Dr. Brinkley in 1823, by whom, under the title “Caustics” it was presented in 1824 to the Royal Irish Academy.

The transport equations.

$$Pu(x, t) = 0, \quad u(x, 0) = \rho(x)e^{i\frac{\psi(x)}{\hbar}}\hbar^{-\ell} \quad \frac{\partial^i}{\partial t^i}u(x, 0) = 0, \quad i = 1, \dots, k-1. \quad (1.1)$$

$$u(x, t) = a(x, t, \hbar)e^{i\phi(x, t)/\hbar} \quad (1.2)$$

$$a(x, t, \hbar) = \hbar^{-\ell} \sum_{i=0}^{\infty} a_i(x, t)\hbar^i. \quad (1.3)$$

Let us return to our project of looking for solutions of the form (1.2) to the partial differential equation and initial conditions (1.1). Our first step was to find the Lagrangian manifold $\Lambda = \Lambda_\phi$ which gave us, locally, a solution of the eikonal equation (1.6). This determines the “phase function” ϕ up to an overall additive constant, and also guarantees that no matter what a'_i s enter into the expression for u given by (1.2) and (1.3), we have

$$Pu = O(\hbar^{-k-\ell+1}).$$

The next step is obviously to try to choose a_0 in (1.3) such that

$$P\left(a_0e^{i\phi(x, t)/\hbar}\right) = O(\hbar^{-k+2}).$$

The next step is obviously to try to choose a_0 in (1.3) such that

$$P\left(a_0 e^{i\phi(x,t)/\hbar}\right) = O(\hbar^{-k+2}).$$

In other words, we want to choose a_0 so that there are no terms of order \hbar^{-k+1} in $P\left(a_0 e^{i\phi(x,t)/\hbar}\right)$. Such a term can arise from three sources:

1. We can take the terms of degree $k-1$ and apply all the differentiations to $e^{i\phi/\hbar}$ with none to a or to ϕ . We will obtain an expression C similar to the principal symbol but using the operator Q obtained from P by eliminating all terms of degree k . This expression C will then multiply a_0 .
2. We can take the terms of degree k in P , apply all but one differentiation to $e^{i\phi/\hbar}$ and the remaining differentiation to a partial derivative of ϕ . The resulting expression B will involve the second partial derivatives of ϕ . This expression will also multiply a_0 .

1. We can take the terms of degree $k - 1$ and apply all the differentiations to $e^{i\phi/\hbar}$ with none to a or to ϕ . We will obtain an expression C similar to the principal symbol but using the operator Q obtained from P by eliminating all terms of degree k . This expression C will then multiply a_0 .
2. We can take the terms of degree k in P , apply all but one differentiation to $e^{i\phi/\hbar}$ and the remaining differentiation to a partial derivative of ϕ . The resulting expression B will involve the second partial derivatives of ϕ . This expression will also multiply a_0 .
3. We can take the terms of degree k in P , apply all but one differentiation to $e^{i\phi/\hbar}$ and the remaining differentiation to a_0 . So we get a first order differential operator

$$\sum_{i=1}^{n+1} A_i \frac{\partial}{\partial x_i}$$

applied to a_0 . In the above formula we have set $t = x_{n+1}$ so as to write the differential operator in more symmetric form.

So the coefficient of \hbar^{-k+1} in $P(a_0 e^{i\phi(x,t)/\hbar})$ is

$$(Ra_0) e^{i\phi(x,t)/\hbar}$$

where R is the first order differential operator

$$R = \sum A_i \frac{\partial}{\partial x_i} + B + C.$$

We will derive the explicit expressions for the A_i , B and C below.

The strategy is to look for solutions of the first order homogenous linear partial differential equation

$$Ra_0 = 0.$$

This is known as the **first order transport equation**.

Having found a_0 , we next look for a_1 so that

$$P\left((a_0 + a_1\hbar)e^{i\phi/\hbar}\right) = O(\hbar^{-k+3}).$$

From the above discussion it is clear that this amounts to solving an inhomogeneous linear partial differential equation of the form

$$Ra_1 = b_0$$

where b_0 is the coefficient of $\hbar^{-k+2}e^{i\phi/\hbar}$ in $P(a_0e^{i\phi/\hbar})$ and where R is the same operator as above. Assuming that we can solve all the equations, we see that we have a recursive procedure involving the operator R for solving (1.1) to all orders, at least locally - up until we hit a caustic!

Geometric re-interpretation of the transport operator.

We will find that when we regard P as acting on $\frac{1}{2}$ -densities (rather than on functions) then the operator R has an invariant (and beautiful) expression as a differential operator acting on $\frac{1}{2}$ -densities on Λ . In fact, the differentiation part of the differential operator will be given by the vector field v_H which we know to be tangent to Λ . The differential operator on Λ will be defined even at caustics. This fact will be central in our study of global asymptotic solutions of hyperbolic equations.

In the next section we shall assume only the most elementary facts about $\frac{1}{2}$ -densities - the fact that the product of two $\frac{1}{2}$ -densities is a density and hence can be integrated if this product has compact support. Also that the concept of the Lie derivative of a $\frac{1}{2}$ -density with respect to a vector field makes sense. If the reader is unfamiliar with these facts they can be found with many more details in Chapter 6.