Correction to Pentagonal Numbers Problem

Recall that we guessed that the solution was quadratic so that the nth pentagonal number – call it $p(n)$ – should be

$$p(m) = am^2 + bm + c$$

We get $p(n)$ from $p(n-1)$ by adding $3m-2$, so that the difference is

$$p(m) - p(m-1) = 3m - 2$$

(Note that in class I wrote $p(m+1) - p(m) = 3m - 2$. This is *wrong*, and led to the wrong answer. In fact $p(m+1) - p(m) = 3(m+1) - 2$.) But using our formula

$$p(m) - p(m-1) = am^2 + bm + c - a(m-1)^2 - b(m-1) - c = 2am - a + b$$

Thus those two quantities are equal,

$$2am - a + b = 3m - 2.$$

Since $2am - a + b = 3m - 2$ for all values of m, that means that corresponding coefficients of m must be equal. Thus $2a = 3$, which means $a = \frac{3}{2}$ and $-2 = -a + b$ so $b = -\frac{1}{2}$.

We still need to find c. Right now we have

$$p(m) = \frac{3}{2}m^2 - \frac{1}{2}m + c$$

to find see, we recall that $p(2) = 5$, but by the above formula $p(2) = \frac{3}{2}2^2 - \frac{1}{2}2 + c = 6 - 1 + c$. So $c = 0$. Thus

$$p(m) = \frac{3}{2}m^2 - \frac{1}{2}m = \frac{m(3m-1)}{2}$$

(Note that $p(3) = \frac{3 \cdot 8}{2} = 12$ and $p(4) = \frac{4 \cdot 11}{2} = 22$, so this formula really does work!)