Some of the problems here have been done or alluded to in section. But it's a good idea at least to go through them so that you make sure you understand them. There is an appendix for things that were not mentioned in lecture.

1 Set Theory

1. Prove that the Cantor Set is uncountable.

2 Topology

1. Let G be a topological group and S a subset. Prove that

$$\mathrm{clos}(S) = \bigcup_{U \ni e, U \text{ open}} SU$$

2. Prove that $\mathbb{A}$, the ring of p-adic adeles is locally Hausdorff. If you really want to get the feel of adeles, then prove that addition $+ : \mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{A}$ is continuous.

3. I showed in section that the closure of a set S in a topological space is the set of all limit points of nets in S. This extends a similar result from metric spaces. Now extend the notion of sequential compactness:

   Prove that a topological space is compact if and only if any net has a converging subnet.

4. Let X be a locally compact Hausdorff space. Let $U_{K,\varepsilon} = \{f | \sup_K |f(x)| < \varepsilon\}$ (K compact, $\varepsilon > 0$). Prove that $V_{f,K,\varepsilon} = f + U_{K,\varepsilon}$ is a topology on $\mathcal{C}(X)$. Prove that $\mathcal{C}(X)$ is complete i.e., any Cauchy net converges.

You probably shouldn't focus that much on topology, since most of the focus was on linear algebra, but these are good problems to know anyway.

3 Linear Algebra

1. I did this in section but having it in the review is probably a good idea. Prove that there is a canonical isomorphism $\phi : V^* \otimes W \longrightarrow Hom(V, W)$ by letting $\phi(v^* \otimes w)(u) = (v^*, u)w$ if and only if at least one of V, W is finite dimensional.

2. Let A be a square matrix, $A_{i,j}$ be the cofactor of $a_{i,j}$. Prove that

$$\sum_i (-1)^{i+j} a_{i,j} \det A_{i,k} = \delta_{j,k} \det A$$

3. Let A be a real or complex matrix. Define

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

Prove that this is well defined and for any $S \in GL_n$ we have $e^{SXS^{-1}} = Se^X S^{-1}$.

4. With the definition above prove that $\det e^A = e^{Tr(A)}$, both for the real and for the complex case. Make sure that you are rigorous when you deal with the real case, i.e. need to extend the scalar field in which you are working with tensor products (formally of course).

4 Inner Product Spaces

1. Let $A(t) = (a_{i,j}(t)) \in \mathcal{M}_{n \times n}(\mathbb{R})$ be a symmetric matrix so that for any i, j, $a_{i,j}(t)$ is a continuous function. For any t, let $\lambda_1(t) \leq \ldots \leq \lambda_n(t)$ be the eigenvalues in increasing order. Prove that $\lambda_i(t)$ is a continuous function for any i.

2. Let A_t be defined as above. Assume that $t_1 \leq t \leq t_2$. If A_{t_1} is positive definite and $\det A_t > 0$ for all such t then A_{t_2} is positive definite.

3. Let X be a compact topological space. Consider $\mathcal{C}(X)$ the space of continuous real valued functions defined on X. Note that the last problem from Topology implies that this is complete. Let E be a vector subspace of $\mathcal{C}(X)$ so that $\forall f \in E$, the set $f(X)$ is finite (in $\mathbb{R}$). Prove that E is finite dimensional.

<center>Appendix</center>

1. Cantor Set. For any interval $[a, b]$ define process P to be removing from its center a subinterval of length one third. Process P therefore leaves intervals $[a, (2a+b)/2], [(a+2b)/3, b]$. Start out with $[0, 1]$ and then apply P. For each of the two intervals left apply P again. Proceed this way, each time applying P to the intervals left. Let I_n be the union of the intervals left after n such application of P. In particular each interval in I_n should have length $1/3^n$. Now take the intersection of all the I_n and that is the Cantor Set.

2. A topological group is a group endowed with a topology that makes multiplication and inversion (in the group) continuous.

3. For any p prime, $\mathbb{Z}_p, \mathbb{Q}_p$ are the p-adic completions of $\mathbb{Z}, \mathbb{Q}$. Let P be the set of all primes. For any $S \subset P$ finite define

$$\mathbb{A}_S = \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p$$

   Define $\mathbb{A} = \bigcup \mathbb{A}_S$ saying that $U \subset \mathbb{A}$ is open if and only if $U \int \mathbb{A}_S$ is open in $\mathbb{A}_S$ (endowed with the product topology) for any finite S.

4. A partially ordered set D is called a directed system if for any $x, y \in D$ there is a $z \in D$ so that $x, y \leq z$. If X is a topological space then a net in X is a pair (f, D) where D is a directed system and $f : D \longrightarrow X$ is a function. A subnet of a net (f, D) is a net (g, E) so that there is a funcion $h : E \longrightarrow D$ so that $f = g \circ h$ and for any $n \in D$ there is a $k \in E$ so that $x \geq k \Longrightarrow h(x) \geq n$.

5. The notion of net generalizes that of a sequence. Limits are easily defined. I will give the definition of Cauchyness:

   A net (f, D) is called Cauchy in a vector space if for any open neighborhood U of the origin, there is an m so that for any $x, y \geq m$ we have $f(x) - f(y) \in U$.

6. A matrix A defines a bilinear pairing on $\mathbb{R}^n$ $h(x, y) = x^T A y$. The matrix is called positive definite if the pairing in positive definite.

<center>3</center>