Math 55a: Honors Advanced Calculus and Linear Algebra

Homework Assignment #5 (18 October 2002):
Linear Algebra I: vector space basics

$$\ast$$

—Axler, page 1 [why?]

Some basic problems (mostly from Chapter 1 of the Axler textbook) on vector spaces and their subsets, intersections, and sums. Problem set is due Friday, Oct. 25, at the beginning of class.

1.–6.[1] Solve problems 4 through 14 on pages 19 and 20 of the Axler textbook. (In problem 4, and problems 8 through 13, V is a vector space over an arbitrary field $\mathbf{F}$.) Which if any of these basic results would fail if $\mathbf{F}$ were replaced by $\mathbf{Z}$?

7. The *symmetric difference* $A \triangle B$ of two subsets A, B of a set X is defined to be the subset of X consisting of those elements of X contained in exactly one of A and B.
   i) Prove that the *power set* $2^X := \{A : A \subseteq X\}$ of X becomes a vector space over the two-element field $\mathbf{Z}/2\mathbf{Z}$ by taking its zero element to be the empty set and defining vector addition by $A + B = A \triangle B$. (This is a generalization of the case $X = \{1, 2, \ldots, n\}$, when 2^X is just $(\mathbf{Z}/2\mathbf{Z})^n$; in that case, you may be familiar with $\triangle$ in the guise of "(bitwise) exclusive or", as opposed to the "inclusive or" which corresponds to the union of sets.)
   ii) Let X now be a topological space, and recall a subset $A \subseteq X$ is said to be *clopen* if it is simultaneously closed and open. Prove that the clopen sets in X constitute a subspace of the $(\mathbf{Z}/2\mathbf{Z})$-vector space 2^X defined in part (i).

[1] These 11 problems, plus the question about "vector spaces over $\mathbf{Z}$", are sufficiently small and straightforward compared to our usual fare that I'm counting each as the equivalent of only half of a problem.