**Math 55a: Honors Advanced Calculus and Linear Algebra**

Homework Assignment #9 (18 November 2002):
Linear Algebra V — Tensors, more eigenstuff, and a bit on inner products

*The terms "proper value", "characteristic value", "secular value", and "latent-value" or "latent root" are sometimes used [for "eigenvalue"] by other authors. The latter term is due to Sylvester [Collected Papers III, 562–4] because such numbers are "latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf." We will not adhere to his terminology.*

— N. Dunford, J.T. Schwartz: *Linear Operators, Part I*, pages 606–7.

We begin with some basic problems on tensors and tensor products. Recall that the *rank* of a linear transformation $T : U \to V$ is the dimension of its image $T(U)$. The *rank* of a matrix is the rank of the linear transformation it represents.

1. Let $\{u_i\}_{i=1}^m$ and $\{v_j\}_{j=1}^n$ be bases of the $F$-vector spaces $U$ and $V$, and consider the general element $w = \sum_i \sum_j w_{ij}(u_i \otimes v_j)$ of $U \otimes V$. Prove that $w$ is the sum of $r$ pure tensors if and only if the matrix $(w_{ij})$ has rank at most $r$.

2. Let $V$ be a vector space of finite dimension $n$ over a field $F$. We constructed a linear map, the trace, from $\mathcal{L}(V)$ to $F$. Hence the map from $\mathcal{L}(V) \times \mathcal{L}(V)$ to $F$ taking $(S, T)$ to the trace of $ST$ is bilinear. Prove that it is symmetric. For what $n$ can there exist $S, T \in \mathcal{L}(V)$ such that $ST - TS$ is the identity map? (By comparison, we observed that the operators $d/dz$ and $z$ on the infinite-dimensional space $\mathcal{P} = F[z]$ satisfy $ST - TS = I$.)

Tensors and eigenstuff:

3. Fix $a \in \mathbf{C}$, and let $T : \mathbf{C} \to \mathbf{C}$ be the map $z \mapsto az$. This is an $\mathbf{R}$-linear operator, so we may consider the linear operator $T' = T \otimes 1$ on the complex vector space $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$. What are the eigenvalues and eigenvectors of $T'$? (Warning: The answer depends on whether $a \in \mathbf{R}$.)

4. Let $U, V$ be vector spaces over a field $F$, equipped with linear operators $S \in \mathcal{L}(U)$, $T \in \mathcal{L}(V)$. Consider $S \otimes T \in \mathcal{L}(U \otimes V)$.
   i) If $\lambda \in F$ is an eigenvalue of $S$, and $\mu \in F$ is an eigenvalue of $T$, prove that $\lambda\mu$ is an eigenvalue of $S \otimes T$.
   ii) If $U, V$ are finite dimensional and $F$ is algebraically closed, prove that every eigenvalue of $S \otimes T$ it the product of an eigenvalue of $S$ with an eigenvalue of $T$.
   iii) Show, by constructing a counterexample with finite-dimensional vector spaces $S, T$ over $\mathbf{R}$, that (ii) no longer holds when the hypothesis on $F$ is dropped.

Apropos eigenstuff... The next result generalizes what we proved in class about involutions (which are the special case $m = 2$, $\lambda_i = \pm 1$).

5. Suppose $V$ is a vector space over a field $F$ and $T$ is a linear operator on $V$ such that $\prod_{i=1}^m (T - \lambda_i I) = 0$ for some *distinct* $\lambda_i \in F$. Prove that $V$ is the direct sum of the $\lambda_i$-eigenspaces of $T$. [NB: $V$ may *not* be assumed finite-dimensional.]

**Tensor products of $A$-modules.** Like direct sums, quotient spaces, and duals, tensor products can be defined in the same way for modules over rings $A$ that need not be fields. Basic properties such as $M \otimes (N \oplus N') \cong (M \otimes N) \oplus (M \otimes N')$ hold in this more general setting, and for much the same reason; but some new phenomena emerge, as in parts (ii) and (iii) of the next problem:

6. i) Show that if $A$ is a commutative ring with unit, and $I \subseteq A$ is an ideal (an additive subgroup such that $aI \subseteq I$ for all $a \in A$, or equivalently a submodule of the $A$-module $A$), then $(A/I) \otimes_A (A/I)$, the tensor product of the quotient $A$-module $A/I$ with itself, is isomorphic with $A/I$.
   ii) On the other hand, show that $(\mathbf{Z}/2\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/3\mathbf{Z})$ is the trivial $\mathbf{Z}$-module $\{0\}$.
   iii) For positive integers $m, n$, what is the $\mathbf{Z}$-module $(\mathbf{Z}/m\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z})$?

Finally, a bit about inner products:

7. Solve Exercises 7 and 13 on pages 122, 123 of Axler. For #13, $V$ is either a real or complex inner-product space, which need not be finite dimensional.

8. Is the symmetric bilinear pairing constructed in Problem 2 nondegenerate? When $F = \mathbf{R}$, is it positive definite?

Axler's exercise #7, as well as the more familiar #6, is often referred to as the "polarization identity". This shows that a linear transformation preserves the norm if and only if it preserves the inner product [more precisely, it shows the harder, "only if" part of this result]. These are basically also the identities used to prove Propositions 2 and 4 in the next chapter (pages 129, 130).

This problem set is due Wednesday [sic], 27 November, at the beginning of class.