Math 55a: Honors Advanced Calculus and Linear Algebra

QUIZ (18 October 2002)

Don't panic![1] These problems should take you considerably less time than the 50-minute class, and the quiz counts for only 5% of your grade.

1. Let d_1, d_2, d_3 be the following three metrics on $\mathbf{R}$: d_1 is the standard metric $d_1(x, y) = |x - y|$; d_2 is the discrete metric; and d_3 is the metric $d_1(x, y) = |x^3 - y^3|$. The identity function $x \mapsto x$ on $\mathbf{R}$ then gives rise to six functions between the metric spaces $(\mathbf{R}, d_1)$, $(\mathbf{R}, d_2)$, $(\mathbf{R}, d_3)$.

   i) Which of these six functions

$$(\mathbf{R}, d_1) \rightleftarrows (\mathbf{R}, d_2), \quad (\mathbf{R}, d_2) \rightleftarrows (\mathbf{R}, d_3), \quad (\mathbf{R}, d_3) \rightleftarrows (\mathbf{R}, d_1)$$

   are continuous?

   ii) Of those, which are uniformly continuous?

   iii) For which $i, j \in \{1, 2, 3\}$ does there exist a subset $S \subseteq \mathbf{R}$ that becomes compact in $(\mathbf{R}, d_i)$ but not in $(\mathbf{R}, d_j)$? Give and justify an example of such an S.

2. Let X be our metric space $\mathcal{C}([0, 1], \mathbf{R})$ of continuous functions on $[0, 1]$ with $d_X(f, g) = \max_{0 \le x \le 1} |f(x) - g(x)|$.

   i) Find an infinite set $S \subset X$ such that the restriction of d_X to S is the discrete metric.

   ii) Let $\mathbf{0} \in X$ be the zero function. Is the closed unit ball $\overline{B}_1(\mathbf{0})$ in X compact? Why?

[1] D. Adams, *The Hitchhiker's Guide to the Galaxy.*