Math 55 Problem Set 3
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1. (i) $d_1(f,f) = \int_0^1 |f(x) - f(x)|\,dx = \int_0^1 0\,dx = 0$. And, if $f \neq g$, then $F(x) = |f(x) - g(x)|$ is not identically zero. Hence $\exists x_0$ so $F(x_0) = 2\epsilon > 0$. And by continuity, $\exists\delta$ such that $\forall x, x_0 - \delta < x < x_0 + \delta, F(x) > \epsilon$. So, $d_1(f,g) = \int_0^1 F(x)\,dx = \int_0^{x_0-\delta} F(x)\,dx + \int_{x_0-\delta}^{x_0+\delta} F(x)\,dx \int_{x_0+\delta}^1 F(x)\,dx \geq \int_{x_0-\delta}^{x_0+\delta}, dx = 2\epsilon\delta > 0$

   $0 = d_1(f,g) = \int_0^1 |f(x) - g(x)|\,dx$, implies $|f(x) - g(x)| = 0$ for all x, and thus $f = g$.

   (ii) $d_1(f,g)$ is symmetric in definition, as $|f - g| = |g - f|$, and thus equals $d_1(g,f)$.

   (iii) $d_1(f,h) + d_1(h,g) = \int_0^1 |f(x) - h(x)| + |h(x) - g(x)|\,dx \geq \int_0^1 |f(x) - g(x)|\,dx = d_1(f,g)$.

2. $\{f_n\}$: $\forall\epsilon > 0, \exists N(> 1/\epsilon)$ so that $\forall n > N, x \in \mathbb{R}$ we have $0 < \frac{n}{x^2+n^2} \leq \frac{n}{n^2} < \epsilon$. Thus, $\{f_n\}$ converges uniformly (and thus also point-wise) to 0 on all of $\mathbb{R}$.

   $\{g_n\}$: For any given x, we can pick a sufficiently large $N(> \sqrt{\frac{x^2}{\epsilon}})$ so that, $n > N$ implies g_n is ϵ close to, 1. Thus, $\{g_n\}$ converges point-wise. But, for any n, if we pick $x = n$ to get $g_n = \frac{1}{2}$. This means that $\{g_n\}$ does not converge uniformly to 1.

3. Let $U = \{x : d(x,A) < d(x,B)\}$ and V defined anaglgously, with d the distance function defined in problem set 2 problem 1. Clearly, U and V are disjoint and contain A and B respectivly. Now, it remains to show that they are open. Let $x \in U$, $\epsilon = \frac{1}{3}(d(x,B) - d(x,A))$, and $y \in B_\epsilon(x)$. Then, $d(y,B) - d(y,A) > d(x,B) - d(x,A) - 2\epsilon > 0$, and we have $y \in U$ and U open. So U and V are as desired.

4. I claim that for U, V open, $U \subset \bar{U} \subset V$, there exists W open so that $\bar{U} \subset W \subset \bar{W} \subset V$. To see this, consider $\bar{U}$ and $X - V$, disjoint closed sets. Then there exists $W \supset \bar{U}$, $W' \supset X - V$ open and disjoint. But then $\bar{U} \subset W \subset X - W' \subset X - V$. Since $X - W'$ is closed and contains W, it also contains $\bar{W}$. And we have $U \subset \bar{U} \subset W \subset \bar{W} \subset V$, as desired.

Now, letting $S_1 \subset X - B$, and $S_0 \supset A$ as produced by $X - B$ and Int$A$ with the above lemma applied twice. I inductivly create open $S_{k/2^i}$ one level of "i" at a time. At each point if $q < r$, then $\bar{S}_q \subset S_r$. $S_{k/2^i}$ (k odd), is generated by the above lemma between $S_{(k-1)/2^i}$ and $S_{(k+1)/2^i}$. So of course, all the sets contain the closure of S_0 and are contained in S_1.

I define $f(x) = \inf(\{1\} \cup \{r : x \in S_r\})$. This is set is bounded below and non empty, so the inf exists. It is clear that $f(x) = 0$ if $x \in A$, $f(x) = 1$ if $x \in B$, and has range $[0, 1]$. It now remains to show that f is continuous. I first show $f(x) = \sup(\{0\} \cup \{r : x \in X - \bar{S}_r\})$. In doing this, we only need to consider r in the sets that are terminating binary fractions, as for no other r is S_r defined and thus is it possible for r to satisfy the condition to be in the sets.

If, $r > f(x)$ then $\exists r_0, r > r_0 > f(x)$ with $x \in S_{r_0} \subset S_r \subset \bar{S}_r$, making $x \notin X - \bar{S}_r$. As $0 \leq f(x)$, we can upperbound the sup with $f(x)$. And, $\forall r < f(x) \leq 1$, then $\exists r_0, r < r_0 < f(x)$ with $x \notin S_{r_0} \supset \bar{S}_r$ and thus $x \in X - \bar{S}_r$. Since all binary fraction $0 < r < f(x)$ (it is key here that $f(x)$ is bounded above 1 so that all of these r produce valid S_r) are in the set, and these are dense in the reals, we have $f(x)$ as a lower bound for the sup as well. This fails if $f(x) = 0$, but then, the additional 0 element saves us. Regardless we have now proved the identity.

Next, we show $f^{-1}([0, r))$ is open (r is now any real number). I claim $f^{-1}([0, r)) = \cup_{s<r} S_s$. If $x \in \cup_{s<r} S_s$, then $\exists s < r$ such that $x \in S_s$ and thus, $f(x) = \inf(\{1\} \cup \{a : x \in S_a\}) \leq s < r$. If $x \notin \cup_{s<r} S_s$, then $\forall s < r, x \notin S_s$. Thus, if $x \in S_s$, then $s \geq r$; and of course $1 \geq r$. So, $f(x) = \inf(\{1\} \cup \{a : x \in S_a\}) \geq r$. So $x \in \cup_{s<r} S_s$ if and only if $x \in f^{-1}([0, r))$ making the sets equal; as the union of open sets is open so is $f^{-1}([0, r))$. The same argument, using the sup definition of f, shows that $f^{-1}((r, 1])$ is open. As, $f^{-1}((a, b)) = f^{-1}([0, b)) \cap f^{-1}((a, 1])$, this set is open as well. Finally, any open set in the reals is the union of open balls (one around each point if you like), f^{-1} of any open set is the union of open sets and is thus open. Thus, f is continuous as desired.

5. Let X be a topological space and $\mathcal{F}$ a collection of closed subsets. Define $\mathcal{G}$ as the complements of sets in $\mathcal{F}$. Then a union of sets in $\mathcal{G}$ is the complement of an intersection of the corresponding sets in $\mathcal{F}$. So

$\mathcal{F}$ has FIP if and only if $\mathcal{G}$ has no finite subcover of X. Also $\mathcal{F}$ has the total intersection property if and only if $\mathcal{G}$ does not cover X. Therefore we are done.

6. Let S be a sequentially compact subset of a metric space.

   (i) Suppose S is not closed. Then $\exists x \notin S$, such that $\forall r > 0, B_r(x) \cap S \neq \emptyset$. Let, a_n be a point in $S \cap B_{1/n}(x)$. Clearly, $a_n \to x \notin S$, and thus all subsequences converge to $x \notin S$ and thus do not converge in S, contradicting the sequential compactness of S.

   (ii) Suppose S is not totally bounded. Then let r be a radius such that there is no r net of S. Now define a_n as follows: Let a_0 be any point in S and let $a_n \in S$, such that $\forall m < n$, $d(a_m, a_n) \geq r$. Such an a_n exists by the lack of an r net of S. But S sequentially compact implies $\exists m, n$ such that $d(a_m, a_n) < r$ which is impossible. So, S must be totally bounded.

7. Clearly if E is totally bounded, then it is totally bounded relative to any metric space containing it. Conversely, if E is totally bounded relative to X, then there is some finite set of point $p_1, p_2, \ldots p_n \in X$, so that every point in E is within $\epsilon/2$ of these points. Without loss of generality, we may assume that there is some $q_i \in E$ in each of these $\epsilon/2$ balls, or else we omit the corresponding p_i from the original enumeration. But now, $\cup_i B_\epsilon(q_i) \supset \cup_i B_{\epsilon/2}(p_i) \supset E$, by the triangle inequality. And thus for every ϵ we have an epsilon net of E centered around points in E, making E totally bounded.

8. Suppose no such r exists. Then for each r, in particular for each $1/n$, there is some x_n such that $\forall \alpha, B_{1/n}(x_n) \not\subseteq U_\alpha$. Now, X is compact, so exists $n_i \in \mathbb{Z}^+, x \in X$, so that $x_{n_i} \to x$. Now, $\{U_\alpha\}$ covers X, so $\exists \alpha$, so that $x \in U_\alpha$. As these sets are open, $\exists r$ such that $B_r(x) \subset U_\alpha$. And, by convergence, $\exists k > \frac{2}{r}$, so that $d(x_k, x) < \frac{r}{2}$. But then, $B_{1/k}(x_k) \subset B_r(x) \subset U_\alpha$, contradicting our construction of $\{x_n\}$, and implying the existence of such an r.