Math 55a: Honors Advanced Calculus and Linear Algebra

Metric topology I: basic definitions and examples

Definition. Metric topology is concerned with the properties of and relations among *metric spaces*. In general, "space" is used in mathematics for a set with a specific kind of structure; in Math 55 we'll also encounter vector spaces, function spaces, inner-product spaces, and more. The structure that makes a set X a metric space is a *distance* d, which we think of as telling how far any two points $p, q \in X$ are from each other. That is, d is a function from $X \times X$ to $\mathbf{R}$. [NB: This is often indicated by the notation $d : X \times X \to \mathbf{R}$; to emphasize that d is a function of two variables one might write $d = d(\cdot, \cdot)$. Cf. the old symbol "$\div$" for division!] The *axioms* that formalize the notion of a distance are:

Nonnegativity: $d(p, q) \geq 0$ for all $p, q \in X$. Moreover, $d(p, p) = 0$ for all p, and $d(p, q) > 0$ if $p \neq q$.
Symmetry: $d(p, q) = d(q, p)$ for all $p, q \in X$.
Triangle inequality: $d(p, q) \leq d(p, r) + d(r, q)$ for all $p, q, r \in X$.

Any function satisfying these three properties is called a *distance function*, or *metric*, on X. A set X together with a metric becomes a *metric space*. Note that strictly speaking a metric space is thus an ordered pair (X, d) where d is a distance function on X. Usually we'll simply call this space X when d is understood.

Examples. The prototypical example of a metric space is $\mathbf{R}$ itself, with the metric $d(x, y) := |x-y|$. (Check that this in fact satisfies all the axioms required of a metric.) Two trivial examples are an empty set, and a one-point set $\{x\}$ with $d(x, x) = 0$.

Having introduced a new mathematical structure one often shows how to construct new examples from known ones. For the structure of a metric space, the easiest such construction is to take an arbitrary *subset* Y of a known metric space X, using the same distance function — more formally, the *restriction* of d to $Y \times Y \subset X \times X$. It should be clear that this is a distance function on Y, which thus becomes a metric space in its own right, known as a metric *subspace* of X. So, for instance, the single metric space $\mathbf{R}$ gives as a huge supply of further metric spaces: simply take any subset, use $d(x, y) = |x - y|$ to make it a subspace of X.

Another construction is the *Cartesian product* $X \times Y$ of two known metric spaces X, Y. This consists of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. There are several choices of metric on $X \times Y$, of which the simplest is the *sup metric* defined by

$$d_{X \times Y}\big((x, y), (x', y')\big) = \max(d_X(x, x'), d_Y(y, y')). \tag{1}$$

(Note that we use subscripts to distinguish the distance functions on X, Y, and $X \times Y$.) So, for instance, taking $X = Y = \mathbf{R}$ we obtain a new metric space

$\mathbf{R} \times \mathbf{R}$, otherwise known as $\mathbf{R}^2$. [Warning: the sup metric

$$d\big((x,y),(x',y')\big) = \max(|x-x'|,|y-y'|)$$

on $\mathbf{R}^2$ is *not* the Euclidean metric you are familiar with. It is a bit tricky to prove analytically that the Euclidean metric satisfies the triangle inequality; we shall do this when we study inner-product spaces a few weeks hence. For an alternative proof, see Rudin, Thms. 1.35 and 1.37e (pages 15,17).] Of course any subset of $\mathbf{R}^2$ then becomes a metric space as well. We can also iterate the product construction, obtaining for instance the metric spaces $\mathbf{R}^2 \times \mathbf{R}$ and $\mathbf{R} \times \mathbf{R}^2$.

Shouldn't both of these simply be called $\mathbf{R}^3$? True, both sets can be identified with ordered triples (x, y, z) of real numbers, arising as $((x, y), z)$ in $\mathbf{R}^2 \times \mathbf{R}$ and as $(x,(y, z))$ in $\mathbf{R} \times \mathbf{R}^2$. But we haven't defined a metric on the Cartesian product $X \times Y \times Z$ of three metric spaces X, Y, Z, and meanwhile we have two metrics coming from the definition (1): one from $(X \times Y) \times Z$, the other from $X \times (Y \times Z)$. Fortunately the two metrics coincide: both tell us that the distance between (x, y, z) and (x', y', z') should be

$$\max(d_X(x,x'), d_Y(y,y'), d_Z(z,z')).$$

In other words, the function $i : (X \times Y) \times Z \to X \times (Y \times Z)$ taking $((x, y), z)$ to $(x,(y, z))$ is not only a bijection of sets but an *isomorphism* of metric spaces, a.k.a. an *isometry*. In general an *isometry* is a bijection $i : X \to X'$ between metric spaces such that $d_X(p, q) = d_{X'}(i(p), i(q))$ for all $p, q \in X$. This definition captures the notion that X, X' are "the same" metric space, and i effects an identification between X, X'. This justifies our identification of $(X \times Y) \times Z$ with $X \times (Y \times Z)$ *as metric spaces*, and calling them both $X \times Y \times Z$. In particular, we have a natural isometry between $\mathbf{R}^2 \times \mathbf{R}$ and $\mathbf{R} \times \mathbf{R}^2$ and may call them both $\mathbf{R}^3$. Likewise we may inductively construct $\mathbf{R}^n$ $(n = 2, 3, 4, \ldots)$ as $\mathbf{R}^m \times \mathbf{R}^{n-m}$ for any integer m with $0 < m < n$; the choice does not matter, because we always get the same distance function

$$d\big((x_1,x_2,\ldots,x_n),(x'_1,x'_2,\ldots,x'_n)\big) = \max_{1\le i\le n} |x_i - x'_i|.$$

We shall give several further basic constructions of metric spaces and examples of isometries in the first problem set.

Bounded metric spaces and function spaces. Perhaps the simplest property a metric space might have is boundedness. A metric space X is said to be *bounded* if there exists a real number B such that $d(p,q) < B$ for all $p,q \in X$. Note that this is not quite the definition given by Rudin (2.18i, p.32). However, the two definitions are equivalent by the following easy

Proposition. *Let E be a nonempty subset of a metric space X. The following are equivalent:*
i) E, considered as a subspace of X, is bounded.
ii) There exists $p \in E$ and a real number M such that $d(p,q) < M$ for all $q \in E$.
iii) There exists $p \in X$ and a real number M such that $d(p,q) < M$ for all $q \in E$.

Proof: We show (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i). (i)$\Rightarrow$(ii) is clear: let $M = B$, and choose for p an arbitrary point of E. The implication (ii)$\Rightarrow$(iii) is even easier, for we may use the same M, p. Finally (iii)$\Rightarrow$(i) is a consequence of the triangle inequality, with $B = 2M$: for $q, q' \in E$ we have $d(q,q') \leq d(p,q) + d(p,q') < M + M = 2M = B$. $\square$

Why did we require E to be nonempty? Note that the empty metric space is bounded by our definition. It is also bounded by Rudin's definition, except when regarded as a subset of the empty metric space — which is surely an oversight. We shall always regard $\emptyset$ as bounded regardless of where we found it, even if nowhere!

Further examples: a finite metric space is bounded; so is an interval $[a,b] := \{x \in \mathbf{R} : a \leq x \leq b\}$, considered as a subspace of $\mathbf{R}$. If X is bounded then so is any subspace; if X, Y are bounded, so is $X \times Y$. The metric space $\mathbf{R}$ is *not* bounded. If X, Y are metric spaces, and X is not bounded, then neither is $X \times Y$, unless Y is empty. (Verify all these!)

Given a bounded metric space X and any set S we may construct a new kind of metric space, a *function space*. We shall call it X^S. As a set, this is simply the space of functions $f : S \to X$. (Do you see why we use the notation X^S for this?) To make it a metric space we define the distance between two functions f, g by

$$d_{X^S}(f,g) := \sup_{s \in S} d_X(f(s), g(s)).$$

[NB this doesn't quite work when $S = \emptyset$; what is X^S then, and what goes wrong with the definition of $d_{X^\emptyset}$? How should we fix it?] This makes sense because $d_X(f(s), g(s)) < B$ for all s, so the set $\{d_X(f(s), g(s)) : s \in S\}$ is bounded and thus has a least upper bound. (See Rudin, Chapter 1 to review this notion if necessary.) Verify that this is in fact a metric space. Actually this is not an entirely new example, since if S is the finite set $\{1, 2, \ldots, n\}$ the space X^S is isometric with X^n. The more general X^S will be a starting point for many important constructions later in the course.