Math 55a: Honors Advanced Calculus and Linear Algebra

Practice Problems — 19 December $\binom{14}{5}$

1. [Contraction mapping theorem; cf. the last problem of the Topology IV set.] A function f from a metric space X to itself is said to be a contraction if there exists a constant $c < 1$ such that $d(f(x), f(y)) \le c d(x, y)$ for all $x, y \in X$ [i.e., f shrinks all distances by a factor of at least $1 : c$]. Prove that f is continuous, and that it has at most one fixed point, i.e., there is at most one $z \in X$ such that $f(z) = z$. Give an example of a nonempty X and a contraction map on X without a fixed point. Show that if X is nonempty and complete then every contraction map has a (necessarily unique) fixed point.

2. For $F = \mathbf{R}$ or $\mathbf{C}$, let $V \subset F^\infty$ be the set of sequences $(a_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty |a_n|^2$ converges (i.e., such that the sequence $\{\sum_{n=1}^N |a_n|^2\}_{N=1}^\infty$ converges in $\mathbf{R}$).
   i) Prove that V is a vector subspace of F^∞.
   ii) Prove that
   $$\langle a, b \rangle := \sum_{n=1}^\infty a_n \overline{b_n} \left[:= \lim_{N\to\infty} \sum_{n=1}^N a_n \overline{b_n} \right]$$
   exists for all $a, b \in V$, and gives an inner product on V.
   iii) Prove that V is complete under the norm associated to this inner product, and is the completion of $\oplus_{n=1}^\infty F$ (with inner product on $\oplus_{n=1}^\infty F$ given by the same formula).
   iv) Prove that the "Hilbert cube"
   $$\{a \in V : |a_n| \le \frac{1}{n} \ (n = 1, 2, 3, \ldots)\}$$
   is a compact subset of V.

3. Let F be the three-element field $\mathbf{Z}/3\mathbf{Z}$, and let a_i, b_i be nonzero elements of F $(1 \le i \le n)$. Prove that the pairings $\langle x, y \rangle_a = \sum_{i=1}^n a_i x_i y_i$ and $\langle x, y \rangle_b = \sum_{i=1}^n b_i x_i y_i$ on F^n are equivalent under $\mathrm{GL}_n(F)$ if and only if $\prod_{i=1}^n a_i = \prod_{i=1}^n b_i$. What happens over $\mathbf{Z}/5\mathbf{Z}$? $\mathbf{Z}/7\mathbf{Z}$? Can you prove a generalization to an arbitrary finite field of odd characteristic?

4. The *Hamilton quaternions* $\mathbf{H}$ are a 4-dimensional real vector space with basis $\{1, i, j, k\}$ and a bilinear product $\mathbf{H} \times \mathbf{H} \to \mathbf{H}$, $(x, y) \mapsto xy$, defined by: $1x = x1 = x$ for all $x \in \mathbf{H}$, and $ij = -ji = k, jk = -kj = i, ki = -ik = j$. *Conjugation* on $\mathbf{H}$ is the linear map: $\mathbf{H} \to \mathbf{H}$, $x \mapsto \bar{x}$, defined by
   $$\overline{a + bi + cj + dk} = a - bi - cj - dk \quad (a, b, c, d \in \mathbf{R}).$$
   The *absolute value* of a quaternion $x = a + bi + cj + dk$ is defined by
   $$|x| = \sqrt{x\bar{x}} = \sqrt{\bar{x}x} = \sqrt{a^2 + b^2 + c^2 + d^2}$$
   (check that these three expressions are in fact always the same). Prove that:
   i) Multiplication in $\mathbf{H}$, though not commutative, satisfies the other standard axioms $x(yz) = (xy)z$ and $x(y + z) = xy + xz$.
   ii) Conjugation is an anti-involution on $\mathbf{H}$, that is, that $\overline{xy} = \bar{y}\,\bar{x}$ for all $x, y \in \mathbf{H}$.
   iii) $|xy| = |x||y|$ for all $x, y \in \mathbf{H}$.
   iv) Every nonzero $x \in \mathbf{H}$ has a unique two-sided multiplicative inverse x^{-1}.
   v) The topological group $x \in \mathbf{H} : |x| = 1$ (with quaternionic multiplication as the group operation) is isomorphic with the group SU_2 of 2×2 complex unitary matrices of determinant 1.

5. Much of our theory of vector spaces, bases, linear transformations, duality, inner products, etc. works for "vector spaces over $\mathbf{H}$." How much of this theory can you obtain?

6. Let V be the three-dimensional real subspace of $\mathbf{H}$ generated by i, j, k. Show that if q is a nonzero quaternion then the map $c_q : x \mapsto qxq^{-1}$ takes V to V. Conclude that $q \mapsto c_q$ is a homomorphism from the multiplicative group $\mathbf{H}^*$ of nonzero quaternions to $O(V)$. What are the kernel and image of this map?