Math 55a, Assignment #10, November 21, 2003

Notations. $\mathbb{R}$ is the field of all real numbers. $\mathbb{C}$ is the field of all complex numbers. $\mathbb{N}$ is the set of all natural numbers (*i.e.,* all positive integers).

Problem 1. Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$. Let V be a Hilbert space over $\mathbb{F}$ whose norm is denoted by $\|\cdot\|$. Denote by B the closed unit ball

$$\left\{ v \in V \,\middle|\, \|v\| \leq 1 \right\}$$

in V. Denote by S the unit sphere

$$\left\{ v \in V \,\middle|\, \|v\| = 1 \right\}$$

in V. Prove that the following three statements are equivalent.

(a) B is compact.

(b) S is compact.

(c) V is finite-dimensional.

(*Hint:* For $(b) \Rightarrow (c)$ and an orthonormal set of elements

$$v_1, v_2, \cdots, v_\ell, \cdots$$

of V consider the open subsets

$$U_j = \left\{ v \in V \,\middle|\, \|v - v_j\| < \frac{1}{\sqrt{2}} \right\}$$

of V for $j \geq 1$.)

Problem 2. Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$. Let V be a Hilbert space over $\mathbb{C}$ whose norm is denoted by $\|\cdot\|$. Let V^* denote the set of all $\mathbb{F}$-valued $\mathbb{F}$-linear functions f on V with

$$\sup \left\{ |f(v)| \,\middle|\, \|v\| \leq 1 \right\} < \infty.$$

Define a collection $\mathcal{T}$ of subsets of V as follows. A subset G of V belongs to T if and only if for every point $v \in G$ there exist $r > 0$ and a finite number of elements $f_1, \cdots, f_k$ of V^* such that the set

$$\left\{ u \in V \,\middle|\, |f_j(u) - f(v)| < r \text{ for } 1 \leq j \leq k \right\}$$

is contained in G (where r and k and $f_1, \cdots, f_k$ of course may depend on the point v of G). The collection $\mathcal{T}$ is known as the *weak topology* of V. Denote by B the closed unit ball

$$\{v \in V \mid \|v\| \leq 1\}$$

in V. Denote by S the unit sphere

$$\{v \in V \mid \|v\| = 1\}$$

in V. Prove that B is compact in the weak topology of V in the sense that if I is any index set and $G_i \in \mathcal{T}$ for $i \in I$ such that $B \subset \bigcup_{i \in I} G_i$, then there exists a finite subset F of I such that $B \subset \bigcup_{i \in F} G_i$. Prove also that S is compact in the weak topology of V. (*Hint:* compare with Part (ii) of Problem 10 in Assignment #2 and consider the image of B under the inclusion $V \hookrightarrow \mathbb{F}^V$ which sends an element v of V to the element of $\mathbb{F}^V$ whose value at $u \in V$ is $\langle u, v \rangle_V \in \mathbb{F}$.)

Problem 3. (Quotient Banach spaces) Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$. Let V be a Banach space over $\mathbb{F}$ with norms $\|\cdot\|_V$. Let W be an $\mathbb{F}$-vector subspace of V which is closed in V. Define the equivalence relation $\sim$ in V so that two v_1 and v_2 of V are equivalent if and only if $v_1 - v_2$ belongs to W. Define the quotient $\mathbb{F}$-vector space V/W as the set of all equivalence classes of this equivalence relation $\sim$. Denote by $\pi : V \to V/W$ the natural projection which maps an element v of V to the equivalence class containing it. For $v \in V$ define $\|\pi(v)\|_{V/W}$ as the infimum of $\|w\|_V$ with $\pi(w) = \pi(v)$.

(a) Prove that V/W with the norm $\|\cdot\|_{V/W}$ is a Banach space over $\mathbb{F}$.

(c) Suppose the Banach space V over $\mathbb{F}$ admits the structure of a Hilbert space over $\mathbb{F}$ in the sense that there is an inner product $\langle \cdot, \cdot \rangle_V$ such that $\|v\|_V{}^2 = \langle v, v \rangle_V$ for $v \in V$. Let U be the orthogonal complement of W in V in the sense that an element of v of V belongs to the subset U of V if and only if $\langle v, w \rangle_V = 0$ for every $w \in W$. Show that the restriction to U of the map $\pi : V \to V/W$ maps U bijectively onto V/W and that $\|u\|_V = \|\pi(u)\|_{V/W}$ for every $u \in U$.

Problem 4. Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$. Let V and W be Hilbert spaces over $\mathbb{F}$ with inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ respectively. For a continuous $\mathbb{F}$-linear map $f : V \to W$, define the *adjoint* map $f^* : W \to V$ of f as the map which sends $w \in W$ to the element $f^*(w)$ of V characterized by $\langle v, f^*(w) \rangle_V = \langle f(v), w \rangle_W$ for all $v \in V$.

(a) Verify that the adjoint map $f^* : W \to V$ of a continuous $\mathbb{F}$-linear map $f : V \to W$ is continuous. Show that a continuous $\mathbb{F}$-linear map $f : V \to W$ is surjective if and only if its adjoint map $f^* : W \to V$ is injective and the image $\operatorname{Im} f^*$ of f^* is a closed subset of V.

(b) An $\mathbb{F}$-linear map $g : V \to W$ is said to be *compact* if for any sequence $\{v_j\}_{j\in\mathbb{N}}$ in V with $\|v_j\|_V \leq 1$ for $j \in \mathbb{N}$ there exists a subsequence $\{v_{j_\nu}\}_{\nu\in\mathbb{N}}$ such that $\{g(v_{j_\nu})\}_{\nu\in\mathbb{N}}$ is a convergent sequence in W. Verify that every compact $\mathbb{F}$-linear map $g : V \to W$ must be continuous. Verify that if $\phi : V \to V$ and $\psi : W \to W$ are continuous $\mathbb{F}$-linear maps and if $g : V \to W$ is a compact $\mathbb{F}$-linear map, then $\psi \circ g \circ \phi : V \to W$ is a compact $\mathbb{F}$-linear map. Show that the adjoint map $g^* : W \to V$ of a compact $\mathbb{F}$-linear map $g : V \to W$ is also compact. (*Hint:* to show the compactness of g^*, for any sequence $\{w_k\}_{k\in\mathbb{N}}$ of elements in the unit ball of W consider the inner product of $(g \circ g^*)\left(w_{k_\lambda} - w_{k_\mu}\right)$ with $w_{k_\lambda} - w_{k_\mu}$ for an appropriate subsequence $\{w_{k_\lambda}\}_{\lambda\in\mathbb{N}}$ of $\{w_k\}_{k\in\mathbb{N}}$.)

(c) Let U be a closed $\mathbb{F}$-vector subspace of V and let $h : U \to V$ be the inclusion map. Let $g : V \to V$ be a compact $\mathbb{F}$-linear map. Show that for any nonzero element λ of $\mathbb{F}$ the kernel $\operatorname{Ker}(g - \lambda h)$ of the $\mathbb{F}$-linear map $g - \lambda h : V \to V$ is finite-dimensional. In other words, the eigenspace of $g : V \to V$ for any nonzero eigenvalue must be finite-dimensional. (*Hint:* use Problem 1.)

Problem 5. (Finite dimensionality of the cokernel of the perturbation of a surjective linear map by a compact linear map) Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$. Let V and W be Hilbert spaces over $\mathbb{F}$. Let f and g be $\mathbb{F}$-linear maps from V to W. Assume that the $\mathbb{F}$-linear map $f : V \to W$ is surjective and continuous and the $\mathbb{F}$-linear map $g : V \to W$ is compact. Show that the image $\operatorname{Im}(f + g)$ is a closed $\mathbb{F}$-vector subspace of W and that the quotient $\mathbb{F}$-vector space $W/\operatorname{Im}(f + g)$ is finite dimensional over $\mathbb{F}$, where $\operatorname{Im}(f+g)$ means the image of the $\mathbb{F}$-linear map $f + g$ from V to W. (*Hint:* use Part (c) of Problem 4 to show that the kernel of $f^* + g^*$ is finite-dimensional.) *Remark:* the term *cokernel* means the quotient of the target space by the image.