Math 55a, Assignment #11, November 28, 2003

Notations. $\mathbb{C}$ denotes the field of all complex numbers. $\mathbb{N}$ denotes the set of all natural numbers (*i.e.,* all positive integers).

Problem 1. (Vandemonde determinant) Let $n \in \mathbb{N}$ and $a_j \in \mathbb{C}$ for $1 \leq j \leq n$ with $a_j \neq a_k$ for $1 \leq j < k \leq n$. Consider the $n \times n$ matrix

$$T = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ (a_1)^2 & (a_2)^2 & (a_3)^2 & \cdots & (a_n)^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ (a_1)^{n-1} & (a_2)^{n-1} & (a_3)^{n-1} & \cdots & (a_n)^{n-1} \end{pmatrix}.$$

(a) By using the simple part of the *Fundamental Theorem of Algebra* that a polynomial of degree m admits no more than m roots with multiplicities counted, show that the determinant $\det T$ of T is nonzero. (The determinant of T is known as the *Vandemonde determinant*).

(b) Now consider each a_j as an independent variable for $1 \leq j \leq n$. By replacing the j-th column of T by the j-column minus the first column for $2 \leq j \leq n$ and expanding the determinant of the resultant matrix according to the first row, show that the determinant $\det T$ of T as a polynomial in the variables $a_1, \cdots, a_n$ is divisible by $a_1 - a_j$ for $2 \leq j \leq n$. Hence show that the determinant $\det T$ of T is equal to

$$\prod_{1 \leq j < k \leq n} (a_k - a_j).$$

(*Hint:* consider $\det T$ as a polynomial of degree $n-1$ in the single variable a_1 and compare the coefficient of $(a_1)^{n-1}$ to the $(n-1) \times (n-1)$ Vandemonde determinant in $a_2, \cdots, a_n$.)

Problem 2. (Homogeneous polynomials and symmetric multilinear functions) Let V be a $\mathbb{C}$-vector space of finite dimension n. Let $e_1, \cdots, e_n$ be a $\mathbb{C}$-basis of V. Let $m \in \mathbb{N}$ and $F(x_1, \cdots, x_n)$ be a $\mathbb{C}$-valued *homogeneous* polynomial of total degree m in the n independent variables $x_1, \cdots, x_n$. (That is,

$F(\lambda x_1, \cdots, \lambda x_n) = \lambda^m F(x_1, \cdots, x_n)$ for all $\lambda \in \mathbb{C}$.) Show that there exists a *unique* $\mathbb{C}$-valued $\mathbb{C}$-multi-linear function

$$G : \underbrace{V \times V \times \cdots \times V}_{m \quad \text{copies}} \longrightarrow \mathbb{C}$$

which is symmetric in its m variables such that

$$G(\underbrace{v, v, \cdots, v}_{m \quad \text{copies}}) = F(a_1, \cdots, a_n)$$

for $v = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$ and $a_j \in \mathbb{C}$ with $1 \leq j \leq n$. (*Hint:* let $\mathcal{A}$ be the $\mathbb{C}$-vector space of all homogeneous polynomials F of degree m and let $\mathcal{B}$ be the $\mathbb{C}$-vector space of all multilinear symmetric functions G on the product of m copies of V. Verify that $\mathcal{A}$ and $\mathcal{B}$ have the same dimension over $\mathbb{C}$ and that the map $\mathcal{B} \to \mathcal{A}$ defined by $G \mapsto F$ is injective, *cf.* Problem 1.)

Problem 3. (Young tableau for three variables) Let V be a $\mathbb{C}$-vector space of finite dimension. For a $\mathbb{C}$-valued $\mathbb{C}$-multi-linear function $f = f(x, y)$ of two variables with $x, y \in V$, there is a decomposition $f(x, y) = g(x, y) + h(x, y)$ into a symmetric function $h(x, y) = h(y, x)$ and a skew-symmetric function $g(x, y) = -g(x, y)$ with

$$h(x, y) = \frac{1}{2}\left(f(x, y) + f(y, x)\right)$$

and

$$g(x, y) = \frac{1}{2}\left(f(x, y) - f(y, x)\right).$$

For the case of a $\mathbb{C}$-valued $\mathbb{C}$-multi-linear function

$$F = F(x_1, x_2, \cdots, x_n)$$

of n variables with $n \geq 3$ and $x_1, \cdots, x_n \in V$, besides the symmetric and skew-symmetric functions a decomposition would involve functions with other type of symmetry properties. Such an additional symmetry property is given by a "Young tableau" which partitions $\{1, 2, \cdots, n\}$ into segments of non-increasing lengths and puts each segment in a row with left justification among all the rows. The projection operator (which sends a general function to a function with the symmetry property) is constructed by summing over all the permutations σ of the variables preserving all the rows and then summing over all the permutations τ of the variables preserving all the columns with coefficients equal to the sign of τ. This problem handles the case of three variables.

For a $\mathbb{C}$-valued $\mathbb{C}$-multi-linear function $f(x_1, x_2, x_3)$ with $x_1, x_2, x_3 \in V$ and a permutation σ of the three numbers $\{1, 2, 3\}$, let

$$(\sigma f)(x_1, x_2, x_3) = f\left(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\right).$$

For $1 \leq i \neq j \leq 3$, the cycle (i, j) denotes the permutation of $\{1, 2, 3\}$ which sends i to j, j to i, and leaves k unchanged if $k \neq i, j$. For $1 \leq i, j, k \leq 3$ distinct, the cycle (i, j, k) is the permutation of $\{1, 2, 3\}$ which sends i to j, j to k, and k to i. Let 1 denote the identity permutation of $\{1, 2, 3\}$.

Let $\mathcal{V}$ denote the set of all $\mathbb{C}$-valued $\mathbb{C}$-multi-linear functions $f(x_1, x_2, x_3)$ with $x_1, x_2, x_3 \in V$. (Note that $\mathcal{V}$ is equal to the tensor product of three copies of the dual vector space V^* of V.) Let

$$\pi_1 = \frac{1}{3}(1 - (1, 3))(1 + (1, 2)) = \frac{1}{3}(1 - (1, 3) + (1, 2) - (1, 2, 3))$$

which is a $\mathbb{C}$-linear map from $\mathcal{V}$ to itself. (Note that π_2 is defined from the Young tableau with the partition of $\{1, 2, 3\}$ into two segments, the first one being $\{1, 2\}$ of length 2 and the second one being $\{3\}$ of length 1.) Similarly, define

$$\pi_2 = \frac{1}{3}(1 - (2, 1))(1 + (2, 3)) = \frac{1}{3}(1 - (2, 1) + (2, 3) - (1, 2, 3))$$

and

$$\pi_3 = \frac{1}{3}(1 - (3, 2))(1 + (3, 1)) = \frac{1}{3}(1 - (3, 2) + (3, 1) - (1, 2, 3))$$

by cyclically permutating $\{1, 2, 3\}$. (Note that 1 is the identity map of $\mathcal{V}$.)

(a) Verify that each π_j satisfies $\pi_j \circ \pi_j = \pi_j$ so that π_j is a projection for $1 \leq j \leq 3$.

(b) Let $\mathcal{V}_{\text{sym}}$ be the $\mathbb{C}$-linear subspace of $\mathcal{V}$ consisting of all $f(x_1, x_2, x_3)$ symmetric in x_1, x_2, x_3. Let π_{sym} be the projection of $\mathcal{V}$ onto $\mathcal{V}_{\text{sym}}$ by averaging σf over all permutations σ of $\{1, 2, 3\}$. Let $\mathcal{V}_{\text{skew}}$ be the $\mathbb{C}$-linear subspace of $\mathcal{V}$ consisting of all $f(x_1, x_2, x_3)$ skew-symmetric in x_1, x_2, x_3. Let π_{skew} be the projection of $\mathcal{V}$ onto $\mathcal{V}_{\text{skew}}$ by averaging (sign of σ) σf over all permutations σ of $\{1, 2, 3\}$. Show that $1 = \pi_{\text{sym}} + \pi_{\text{skew}} + \pi_1 + \pi_2 + \pi_3$ and that $\pi \circ \pi' = 0$ if π and π' are distinct elements of the set $\{\pi_{\text{sym}}, \pi_{\text{skew}}, \pi_1, \pi_2, \pi_3\}$ so that the $\mathbb{C}$-vector space $\mathcal{V}$ is the direct sum of the images of $\mathcal{V}$ under $\pi_{\text{sym}}, \pi_{\text{skew}}, \pi_1, \pi_2, \pi_3$.