Math 55a, Assignment #12, December 5, 2003

Problem 1. (Points with different right-hand and left-hand derivatives countable) Let $-\infty \leq a < b \leq +\infty$ and $f(x)$ be a real-valued function on (a, b). For $x \in (a, b)$ the left-hand derivative $f'_-(x)$ of $f(x)$ at x is defined by

$$f'_-(x) = \lim_{y \to x,\, y < x} \frac{f(y) - f(x)}{y - x}$$

and the right-hand derivative $f'_+(x)$ of $f(x)$ at x is defined by

$$f'_+(x) = \lim_{y \to x,\, y > x} \frac{f(y) - f(x)}{y - x}.$$

Let E be the set of points $x \in (a, b)$ such that both $f'_-(x)$ and $f'_+(x)$ but $f'_-(x) \neq f'_+(x)$. Prove that the set E is at most countable. (*Hint:* consider the subset E_- of points x of E where $f'_-(x) < f'_+(x)$. Arrange all rational numbers in $\{r_j\}_{j \in \mathbb{N}}$. For $x \in E_-$ choose the smallest k such that $f'_-(x) < r_k < f'_+(x)$, the smallest m such that $\frac{f(y)-f(x)}{y-x} < r_k$ for $r_m < y < x$, and the smallest n such that $\frac{f(y)-f(x)}{y-x} < r_k$ for $x < y < r_n$. Verify that the map $x \mapsto (k, m, n)$ is injective.)

Problem 2. (Sufficient Second Derivative Condition for Convexity.) Let $-\infty \leq a < b \leq +\infty$ and $f(x)$ be a real-valued function on (a, b). Suppose $f''(x)$ exists and $f''(x) > 0$ for every $x \in (a, b)$. Verify that the function $f(x)$ is (strictly) convex on (a, b) in the sense that

$$f(x) < \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

for $a < x_1 < x < x_2 < b$. (*Hint:* use the Mean Value Theorem.)

Problem 3. (Newton's Method of Iteration to Find Roots – Problem 25 on Page 118 of Rudin's Book) Suppose f is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f'(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$. Complete the details in the following outline of *Newton's method* for computing ξ.

(a) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Interpret this geometrically, in terms of a tangent to the graph of f.

(b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n\to\infty} x_n = \xi.$$

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)} (x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

(d) If $A = \frac{M}{2\delta}$, deduce that

$$0 \le x_{n+1} - \xi \le \frac{1}{A} \left[A\left(x_1 - \xi\right)\right]^{2^n}.$$

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ξ?

(f) Put $f(x) = x^{\frac{1}{3}}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

Problem 4. (Uniqueness of Solution of Ordinary Differential Equations)

(a) (Problem 26 on Page 119 of Rudin's Book) Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \le A\,|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (*Hint:* Fix $x_0 \in [a, b]$, let $M_0 = \sup |f(x)|$ and $M_1 = \sup |f'(x)|$ for $a \le x \le x_0$. For any such x,

$$|f(x)| \le M(x_0 - a) \le A(x_0 - a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$. Proceed.)

(b) (Problem 27 on Page 119 of Rudin's Book) Let ϕ be a real function defined on a rectangle R in the plane, given by $a \le x \le b$, $\alpha \le y \le \beta$. A *solution* of the initial-value problem

$$y'' = \phi(x, y), \quad y(a) = c \quad (\alpha \le c \le \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \le f(x) \le \beta$, and

$$f'(x) = \phi(x, f(x)) \quad (a \le x \le b).$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \le A\,|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$. (*Hint:* Apply Part (a) to the difference of two solutions. Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{\frac{1}{2}}, \quad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = \frac{x^2}{4}$. Find all other solutions.)

(c) (Problem 28 on Page 119 of Rudin's Book) Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

$$y_j' = \phi_j\left(x, y_1, \cdots, y_k\right), \quad y_j(a) = c_j \quad (j = 1, \cdots, k).$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \vec{\phi}\,(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y} = (y_1, \cdots, y_k)$ ranges over a k-cell, $\vec{\phi}$ is the mapping of a $(k+1)$-cell into the Euclidean k-space whose components are the functions $\phi_1, \cdots, \phi_k$, and $\mathbf{c}$ is the vector $(c_1, \cdots, c_k)$. Use Part (b) for vector-valued functions.

(d) (Problem 29 on Page 119 of Rudin's Book) Specialize Part (c) by considering the system

$$y_j' = y_{j+1} \quad (j = 1, \cdots, k-1),$$
$$y_k' = f(x) - \sum_{j=1}^{k} g_j(x)(x) y_j,$$

where $f, g_1, \cdots, g_k$ are continuous real-valued functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x)y^{k-1} + \cdots + g_2(x)y' + g_1(x)y = f(x),$$

subject to the initial conditions

$$y(a) = c_1,\ y'(a) = c_2,\ \cdots,\ y^{(k-1)}(a) = c_k.$$

(e) Let n be a positive integer. Formulate and prove an analogous uniqueness theorem for systems of higher-order differential equations of the form

$$\begin{gathered} y^{(n)} = \phi_j\left(x, y_1, \cdots, y_k, y_1', \cdots, y_k', \cdots, y_1^{(n-1)}, \cdots, y_k^{(n-1)}\right), \\ y_j(a) = c_j^{(0)}, y_j'(a) = c_j^{(1)}, \cdots, y_j^{(n-1)}(a) = c_j^{(n-1)} \quad (j = 1, \cdots, k) \end{gathered}$$

Note that this can be rewritten in the form

$$\begin{gathered} \mathbf{y}^{(n)} = \vec{\phi}\left(x, \mathbf{y}, \mathbf{y}', \cdots, \mathbf{y}^{(n-1)}\right), \\ \mathbf{y}(a) = \mathbf{c}^{(0)}, \mathbf{y}'(a) = \mathbf{c}^{(1)}, \cdots, \mathbf{y}^{(n-1)}(a) = \mathbf{c}^{(n-1)}, \end{gathered}$$

where $\mathbf{y} = (y_1, \cdots, y_k)$ ranges over a k-cell, $\vec{\phi}$ is the mapping of a $(nk+1)$-cell into the Euclidean k-space whose components are the functions $\phi_1, \cdots, \phi_k$, and $\mathbf{c}$ is the vector $(c_1, \cdots, c_k)$.

Problem 5. (Inequality Form of Taylor's Theorem for Vector-Valued Functions) Suppose $\mathbf{f}$ is a map from $[a, b]$ to $\mathbb{R}^k$, n is a positive integer, $\mathbf{f}^{(n-1)}$ is continuous on $[a, b]$, $\mathbf{f}^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$. Show that there exists a point x between α and β such that

$$\left\| \mathbf{f}(\beta) - \sum_{k=0}^{n-1} \frac{1}{k!} \mathbf{f}^{(k)}(\alpha) (\beta - \alpha)^k \right\| \leq \frac{|\beta - \alpha|^n}{n!} \left\| \mathbf{f}^{(n)}(x) \right\|.$$

Problem 6. (Higher-Order Difference Quotient)

(a) (Problem 11 on Page 115 of Rudin's Book) Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if $f''(x)$ does not. (*Hint:* Use l'Hôpital's rule.)

(b) For a function $g(x)$ define

$$(\Delta_h g)(x) = \frac{g(x+h) - g(x)}{h}.$$

Suppose f is defined in a neighborhood of x, and suppose $f^{(n)}(x)$ exists for some positive integer n. Show that

$$\lim_{h \to 0} \frac{\left((\Delta_h)^n f\right)(x)}{h^n} = f^{(n)}(x).$$

Problem 7. (Fixed Point of Contraction Map – Problem 22 on Page 117 of Rudin's Book) Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

(b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| < A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \cdots$.

(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \cdots.$$

Problem 8. (Mixed-Difference-Quotient-Differential Remainder Term in Taylor's Theorem – Problem 18 on Page 116 of Rudin's Book) Suppose f is a real function on $[a,b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a,b]$. Let α, β be distinct points of $[a,b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t-\alpha)^k \,.$$

Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a,b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t-\beta)Q(t)$$

$n-1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!} (\beta - \alpha)^n \,.$$