Math 55a, Assignment #2, Sept. 26, 2003

Problem 1. Let

$$d(z,w) = \frac{|z-w|}{\sqrt{\left(1+|z|^2\right)\left(1+|w|^2\right)}} \tag{$*$}$$

for z, $w \in \mathbb{C}$.

(i) Verify that, if a, $b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$ and if

$$z' = \frac{az+b}{-\bar{b}z+\bar{a}}$$

and

$$w' = \frac{aw+b}{-\bar{b}w+\bar{a}},$$

then $d(z,w) = d(z',w')$.

(ii) Verify that $d(z,w)$ defined in $(*)$ is a metric of $\mathbb{C}$. (*Hint:* Choose a, b to move a given triple of points in $\mathbb{C}$ to some suitable special positions to facilitate the verification.)

Problem 2. (Problem 17 on Page 44 of Rudin's book) Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense? Is E compact? Is E perfect?

Problem 3. Let A be a nonempty set. Let $X = \mathbb{R}^A$ be the set of all maps $f : A \to \mathbb{R}$ such that $\{|f(a)|\}_{a\in A}$ admits an upper bound in $\mathbb{R}$. For f, $g \in X$ define the distance function $d(f,g)$ by

$$d(f,g) = \sup_{a\in A} |f(a) - g(a)|.$$

Let E be the subset of X consisting of all $f : A \to \mathbb{R}$ such that $|f(a)| \leq 1$ for all $a \in A$. Show that

(a) E is a closed subset of the metric space X;

(b) E is compact if and only if the number of elements in A is finite.

Problem 4. (Problem 23 on Page 45 of Rudin's book) A metric space is called *separable* if it contains a countable dense set. A collection $\{V_\alpha\}_{\alpha\in A}$ of open subsets of X is said to be a *base* for X if for every $x \in X$ and every open subset G of X with $x \in G$ there exists some $\alpha \in A$ such that $x \in V_\alpha \subset G$. Prove that every separable metric space has a *countable* base. (*Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.)

Problem 5. Let I be the closed interval $[-1, 1]$ in $\mathbb{R}$. Let X be the set $I^{\mathbb{N}}$ of all maps $f : \mathbb{N} \to I$. For $f, g \in X$ define the distance function $d(f, g)$ by

$$d(f,g) = \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} \frac{1}{2^k} |f(k) - g(k)|$$

so that X becomes a metric space.

(i) Show that the collection of subsets of X, which are of the form

$$\left\{ f \in X \;\middle|\; d(f(k_j), g(k_j)) < r_j \text{ for } 1 \le j \le \ell \right\}$$

for some $\ell, k_1, \cdots, k_\ell \in \mathbb{N}$ and some $r_1, \cdots, r_\ell \in \mathbb{R}_{>0}$ and some $g \in X$, is a base for X.

(ii) Show that X is separable.

Problem 6. (Problem 24 on Page 45 of Rudin's book) Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. (*Hint:* Firx $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \cdots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \ge \delta$ for $1 \le i \le j$. SHow that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = \frac{1}{n}$ $(n \in \mathbb{N})$ and consider the centers of the corresponding neighborhoods.)

Problem 7. (Problem 26 on Page 45 of Rudin's book) Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. (*Hint:* By the preceding two problems, X has a countable base. It follows that every open cover of X has a *countable* subcover $\{G_n\}_{n\in\mathbb{N}}$. If no finite subcover of $\{G_n\}_{n\in\mathbb{N}}$ covers X, then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each $n \in \mathbb{N}$, but $\cap_{n\in\mathbb{N}} F_n$ is empty. If E is a set which contains a point from each F_n, consider a limit point of E to get a contradiction.)

Problem 8. Let X be the set of all maps $f:\mathbb{N}\to\mathbb{R}$ such that $\left\{\sum_{k=1}^{n}|f(a)|\right\}_{n\in\mathbb{N}}$ admits an upper bound in $\mathbb{R}$. For $f,\,g\in X$ define the distance function $d\,(f,g)$ by

$$d\,(f,g)=\sup_{n\in\mathbb{N}}\sum_{k=1}^{n}|f(k)-g(k)|\,.$$

Let F be the subset of X consisting of all f such that

$$\sum_{k=1}^{n}k\,|f(k)|\le 1\quad\text{for all }n\in\mathbb{N}.$$

Then F is compact. (*Hint:* Use Problem 7.)

Definition of Subbase. A collection $\{W_\beta\}_{\beta\in B}$ of open subsets of X is called a *subbase* for X if the family

$$\{W_{\beta_1}\cap\cdots W_{\beta_k}\}_{k\in\mathbb{N},\beta_1,\cdots,\beta_k\in B}$$

of all finite intersections of members of $\{W_\beta\}_{\beta\in B}$ is a base for X. (See Problem 4 for the definition of a base for X.)

Problem 9. Let X be a metric space and $\{W_\beta\}_{\beta\in B}$ be a subbase for X. Show that X is compact if and only if every cover of X by members of $\{W_\beta\}_{\beta\in B}$ has finite subcover. In other words, X is compact if and only if for any subset C of B with $X=\cup_{\beta\in C}W_\beta$ there exists a finite subset D of C such that $X=\cup_{\beta\in D}W_\beta$. (*Hint:* For the "if" part, it suffices to show that no collection $\{G_\alpha\}_{\alpha\in A}$ of open subsets of X can cover X if it satisfies the condition that

$$(\dagger)\qquad \cup_{\alpha\in E}G_\alpha\ne X\quad\text{for any finite suset }E\text{ of }A.$$

Suppose there is such a collection $\{G_\alpha\}_{\alpha\in A}$ which satisfies (†) and which covers X. First show that we can enlarge A if necessary (by adding more new open subsets of X to the collection) so that $\{G_\alpha\}_{\alpha\in A}$ is *maximal* in the sense that any further enlargement of A results in a new collection violating (†), because the *increasing* union of collections satisfying (†) still satisfies (†). Let A' consist of all $\alpha\in A$ with $G_\alpha\in\{W_\beta\}_{\beta\in B}$. Take $x\in X$ not in $\cup_{\alpha\in A'}G_\alpha$. Then $x\in\cap_{\beta\in L}W_\beta\subset G_{\alpha_0}$ for some finite subset L of B and some $\alpha_0\in A$. By the maximality of $\{G_\alpha\}_{\alpha\in A}$, adding any one single W_β with $\beta\in L$ to the collection $\{G_\alpha\}_{\alpha\in A}$ results in a new collection which violates (†). On the other hand, the condition $\cap_{\beta\in L}W_\beta\subset G_{\alpha_0}$ implies that the addition of any one single W_β with $\beta\in L$ cannot change the satisfaction of (†) to violation for $\{G_\alpha\}_{\alpha\in A}$.)

Problem 10. Let X_n $(n \in \mathbb{N})$ be a compact metric space with metric $d_n(\cdot,\cdot)$. Let a_n be the maximum of 1 and $\sup_{x,y\in X_n} d_n(x,y)$.

(i) Show that $\prod_{n\in\mathbb{N}} X_n$ is a *compact* metric space with the metric

$$d(x,y) = \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} \frac{1}{2^k\, a_k}\, d_k(x_k, y_k),$$

where $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$. (*Hint:* compare with Problem 5 and Part (ii) of this problem and consider the hint in Part (ii) of this problem.)

(ii) More generally, if A is any index set and X_α $(\alpha \in A)$ is a compact metric space with metric $d_\alpha(\cdot,\cdot)$ and if $\mathcal{G}$ denotes the collection of all subsets of $\prod_{\alpha\in A} X_\alpha$ of the form

$$\left\{ x \in \prod_{\alpha\in A} X_\alpha \;\middle|\; d_{\alpha_1}(x,y) < r_1, \cdots, d_{\alpha_k}(x,y) < r_k \right\}$$

for some $k \in \mathbb{N}$ and some $\alpha_1, \cdots, \alpha_k \in A$ and some $y \in \prod_{\alpha\in A} X_\alpha$ and some $r_1, \cdots, r_k \in \mathbb{R}_{>0}$, then any cover of X by elements of $\mathcal{G}$ admits a finite subcover. (*Hint:* apply the argument in Problem 9 by replacing the subbase of Problem 9 by the collection consisting of all subsets of $\prod_{\alpha\in A} X_\alpha$ of the form

$$\left\{ x \in \prod_{\alpha\in A} X_\alpha \;\middle|\; d_{\alpha_0}(x,y) < r_0 \right\}$$

for some $\alpha_0 \in A$ and some $y \in \prod_{\alpha\in A} X_\alpha$ and some $r_0 > 0$.)

Problem 11. (Problem 30 on Page 46 of Rudin's book) Imitate the proof of Theorem 2.43 in Rudin's book to obtain the following result:

If $\mathbb{R}^k = \cup_{n=1}^{\infty} F_n$, where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.

Equivalent statement: If G_n is a dense open subset of $\mathbb{R}^k$ for $n \in \mathbb{N}$, then $\cap_{n=1}^{\infty} G_n$ is not empty (in fact, is dense in $\mathbb{R}^k$).

Problem 12. Suppose X is a metric space and $\{E_\alpha\}_{\alpha\in A}$ is a collection of subsets of X such that each E_α is connected and $X = \cup_{\alpha\in A} E_\alpha$. Suppose for any $\alpha, \beta \in A$ there exist a finite number of elements $\alpha_1, \cdots, \alpha_k \in A$ with $\alpha_1 = \alpha$ and $\alpha_k = \beta$ such that $E_{\alpha_j} \cap E_{\alpha_{j+1}}$ is nonempty for $1 \le j < k$. Show that X is connected.

Problem 13. Prove that the product of a finite number of connected metric spaces is connected. Prove that if E is a subset of a metric space X and if E is connected, then the closure of E in X is connected. Construct an example of a connected subset of $\mathbb{R}^2$ whose interior is not connected.

Problem 14. Let X be a *connected* metric space with metric $d(\cdot,\cdot)$, Y be any set, and $f : X \to Y$ be a map. Suppose that f is *locally constant* in the sense that for every $x \in X$ there exist $y \in Y$ and $r \in \mathbb{R}_{>0}$ (both of which may depend on x) such that $f(z) = y$ for any $z \in X$ with $d(x,z) < r$. Show that f is constant in the sense that there exists some $c \in Y$ such that $f(x) = c$ for all $x \in X$.

Problem 15. If A is a subset of a metric space X, show that at most fourteen different sets can be obtained by repeatedly applying to A the operations of taking closures and taking complements (for example, "the closure of the complement of the closure of A" and "the closure of the complement of the closure of the complement of the closure of A"). Construct a set in $\mathbb{R}$ for which the fourteen different sets actually occur.