Math 55a, Assignment #3, October 3, 2003

Problem 1. (Problem 3 on Page 78 in Rudin's book) If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \quad (n = 1, 2, 3, \cdots),$$

prove that $\{s_n\}$ converges and that $s_n < 2$ for $n = 1, 2, 3, \cdots$.

Problem 2. (Problem 4 on Page 78 in Rudin's book) Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0\,; \quad s_{2m} = \frac{s_{2m-1}}{2}\,; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Problem 3. For any real number x show that

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

Problem 4. (Problem 22 on Page 82 in Rudin's book) Suppose X is a nonempty complete metric space, and $\{G_n\}_{n\in\mathbb{N}}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\cap_{n=1}^{\infty} G_n$ is not empty. (In fact, it is dense in X.) (*Hint:* Find a shrinking sequence of neighborhoods E_n such that $\overline{E_n} \subset G_n$, and apply the statement that, if $\{F_n\}$ is a sequence of closed nonempty and bounded subsets in a complete metric space with $F_n \supset F_{n+1}$ and $\lim_{n\to\infty} \operatorname{diam} F_n = 0$, then $\cap_{n=1}^{\infty} F_n$ consists of exactly one point.)

Problem 5. (Problem 24 on Page 82 in Rudin's book) Let X be a metric space.

(a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X *equivalent* if

$$\lim_{n\to\infty} d\left(p_n, q_n\right) = 0.$$

Prove that this is an equivalent relation.

(b) Let X^* be the set of all equivalent classes so obtained. If $\mathbf{P} \in X^*$, $\mathbf{Q} \in X^*$, $\{p_n\} \in \mathbf{P}$, $\{q_n\} \in \mathbf{Q}$, define

$$\Delta\left(\mathbf{P}, \mathbf{Q}\right) = \lim_{n\to\infty} d\left(p_n, q_n\right).$$

Show that the number $\Delta\left(\mathbf{P}, \mathbf{Q}\right)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence Δ is a distance function in X^*.

(c) Prove that the resulting metric space X^* is complete.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p ; let $\mathbf{P}_p$ be the element of X^* which contains this sequence. Prove that

$$\Delta\left(\mathbf{P}_p, \mathbf{P}_q\right) = d\left(p, q\right)$$

for all p, $q \in X$. In other words, the mapping φ defined by $\varphi(p) = \mathbf{P}_p$ is an isometry (*i.e.,* a distance-preserving mapping) of X into X^*.

(e) Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X.

Problem 6. Let $\gamma > 1$ and let A be any nonempty set. Let X be the subset of the set $A^{\mathbb{Z}}$ of all maps from $\mathbb{Z}$ to A, which is defined as follows. A map $f : \mathbb{Z} \to A$ belongs to X if and only there exists some $\ell \in \mathbb{Z}$ such that $f(n) = 0$ for $n < \ell$. Define the following metric $d_X\left(\cdot, \cdot\right)$ on X. For f, $g \in X$, the distance $d_X\left(f, g\right)$ is equal to $\gamma^{-\ell}$ where ℓ is the largest integer such that $f(n) = g(n)$ for $n < \ell$. Show that X is complete with respect to the metric $d_X\left(\cdot, \cdot\right)$. In other words, every Cauchy sequence in the metric space X has a limit in X.

Problem 7. Let p be a prime number. In Problem 6, let $\gamma = p$ and $A = \{0, 1, , \cdots, p-1\}$ and let X be the metric space constructed in Problem 6 with metric $d_X\left(\cdot, \cdot\right)$. Let Y be the set of all rational numbers of the form $\frac{m}{p^k}$ with m, $k \in \mathbb{Z}$. When m is not divisible by p, let $\left\|\frac{m}{p^k}\right\|_p = p^k$. Define the metric $d_Y\left(\cdot, \cdot\right) = \|a - b\|_p$ for a, $b \in Y$. For any $a \in Y$ there exist uniquely $b_k, b_{k+1}, \cdots, b_\ell \in A$ with $k \le \ell$ in $\mathbb{Z}$ such that $a = \sum_{j=k}^{\ell} b_j p^j$. Define the map $\Phi : Y \to X$ by $\Phi(a) : \mathbb{Z} \to A$ with $\left(\Phi(a)\right)(n) = b_n$ for $k \le n \le \ell$ and $\left(\Phi(a)\right)(n) = 0$ for $n < k$ or $n > \ell$. Show that the map Φ is distance-preserving and that X is the completion of Y (when Y is embedded into X by Φ).

Problem 8. Let p be a prime number and let Y be the set of all rational numbers of the form $\frac{m}{p^k}$ with m, $k \in \mathbb{Z}$. Define a metric $d\left(\cdot, \cdot\right)$ in Y by

$$d\left(a, b\right) = \frac{|a - b|}{1 + |a - b|}.$$

Here $|a-b|$ means the usual absolute value of the difference between a and b as elements of $\mathbb{Q}$. What is the completion of Y? (*Hint:* consider $\mathbb{R}$.)

Problem 9. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of real numbers. Let

$$b_1 \geq b_2 \geq \cdots \geq b_k \geq \cdots \geq 0$$

be a non-increasing sequence of non-negative numbers. Prove that $\sum_{n=1}^{\infty} a_n b_n$ is a convergent series. (*Hint:* use the following discrete analogue of "integration by parts":

$$\begin{gathered} A_1 B_1 + A_2 B_2 + \cdots + A_n B_n \\ = S_1 (B_1 - B_2) + S_2 (B_2 - B_3) + \cdots + S_{n-1} (B_{n-1} - B_n) + S_n B_n, \end{gathered}$$

where $S_n = A_1 + \cdots + A_n$.)

Problem 10. Let a_n be a sequence of complex numbers with $a_n \neq -1$ for each $n \in \mathbb{N}$. Assume that the sequence $\prod_{k=1}^{n} (1 + |a_k|)$ approaches some nonzero real number as its limit as $n \to \infty$. Show that the sequence of complex numbers $\prod_{k=1}^{n} (1 + a_k)$ approaches some nonzero complex number as its limit as $n \to \infty$. (*Hint:* use

$$e^{\alpha_1 + \cdots + \alpha_n} \geq (1 + \alpha_1) \cdots (1 + \alpha_n) \geq \alpha_1 + \cdots + \alpha_n$$

for nonnegative real numbers $\alpha_1, \cdots, \alpha_n$ and consider the convergence behavior of $\sum_{n=1}^{\infty} |a_n|$.)