Math 55a, Assignment #4, October 10, 2003

Problem 1. (Generalization of Cauchy's theorem on convergence of series based on the analogue of change of variables in integrals) Let $a_k \in \mathbb{R}$ $(k \in \mathbb{N})$ be non-increasing and nonnegative, *i.e.,* $0 \leq a_{k+1} \leq a_k$ for $k \in \mathbb{N}$. Let $m_k \in \mathbb{N}$ for $k \in \mathbb{N}$ such that $(m_k)^2 \geq m_{k+1}m_{k-1}$ for $k \in \mathbb{N}$ with $k \geq 2$. Let $N_n = \sum_{k=1}^n m_k$. Show that $\sum_{k=1}^\infty a_k$ converges if and only if $\sum_{k=1}^\infty m_k a_{N_k}$ does. (*Hint:* Compare with Cauchy's theorem that a series of non-increasing nonnegative terms $\sum_{k=1}^\infty a_k$ converges if and only if $\sum_{k=1}^\infty 2^k\, a_{2^k}$ does. Compare the convergence behaviors of $\sum_{k=1}^\infty m_k a_{N_k}$ and $\sum_{k=1}^\infty m_{k+1} a_{N_k}$.)

Definition. A real number α is said to be *algebraic* if there exist $a_0, \cdots, a_n \in \mathbb{Z}$ with $a_n \neq 0$ such that $\sum_{j=1}^n a_j \alpha^j = 0$. A real number is said to be transcendental if it is not algebraic.

Problem 2. (Proof of transcendence by rate of approximation by rational numbers)

(a) Let $n \geq 2$ be an integer and $\alpha \in \mathbb{R}$ and $a_0, \cdots, a_n \in \mathbb{Z}$ with $a_n \neq 0$. Assume that $\sum_{j=1}^n a_j \alpha^j = 0$. If the polynomial $\sum_{j=1}^n a_j x^j$ is in x irreducible (*i.e,* it is not the product of two polynomials of degree $< n$), show that there exists some $C > 0$ such that $\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^n}$ for any $p, q \in \mathbb{Z}$ with $q \neq 0$. (*Hint:* consider the absolute value of both sides of $f(x) - f(\alpha) = (x - \alpha)g(x)$, where $f(x) = \sum_{j=1}^n a_j x^j$ and $g(x)$ is a polynomial which is nonzero at α.)

(b) Show that the real number

$$\alpha = \sum_{k=1}^\infty \frac{1}{2^{k!}}$$

is transcendental. (*Hint:* compare

$$\alpha - \sum_{k=1}^{\ell-1} \frac{1}{2^{k!}}$$

with

$$\frac{1}{(2^{\ell!})^n}$$

with $n \in \mathbb{N}$ and ℓ sufficiently large.)

Problem 3. (Problem 14 on Page 80 of Rudin's book) (Comparison of convergence of a sequence and that of its arithmetic mean) If $\{s_n\}$ is a complex sequence, define its arithmetic mean by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \quad (n = 0, 1, 2, \cdots).$$

(a) If $\lim_{n\to\infty} s_n = s$, prove that $\lim_{n\to\infty} \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim_{n\to\infty} \sigma_n = 0$.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup_{n\to\infty} s_n = \infty$, although $\lim_{n\to\infty} \sigma_n = 0$?

(d) Put $a_n = s_n - s_{n-1}$ for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k\, a_k \ .$$

Assume that $\lim_{n\to\infty} (n a_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a conveser of (a), but under additional assumption that $n a_n \to 0$.

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|n a_n| \leq M$ for all n, and $\lim_{n\to\infty} \sigma_n = \sigma$. Prove that $\lim_{n\to\infty} s_n = \sigma$, by completing the following outline:

If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m} (\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^{n} (s_n - s_i) \, .$$

For these i,

$$|s_n - s_i| \leq \frac{(n-i)\, M}{i+1} \leq \frac{(n-m-1)\, M}{m+2} \ .$$

Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n - \varepsilon}{1 + \varepsilon} < m + 1.$$

Then $(m+1)\,(n-m) \leq \frac{1}{\varepsilon}$ and $|s_n - s_i| < M\,\varepsilon$. Hence

$$\limsup_{s\to\infty} |s_n - \sigma| \leq M\varepsilon.$$

Since ε was arbitrary, $\lim_{n\to\infty} s_n = \sigma$.

Definition. Let Λ be an index set and $\sum_{n=1}^{\infty} c_n^{(\lambda)}$ be a collection of series indexed by $\lambda \in \Lambda$. The collection of series $\sum_{n=1}^{\infty} c_n^{(\lambda)}$ $(\lambda \in \Lambda)$ is said to be *uniformly Cauchy* with respect to $\lambda \in \Lambda$ if given any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ independent of $\lambda \in \Lambda$ such that $\left|\sum_{n=p}^{q} c_n^{(\lambda)}\right| < \varepsilon$ for $q \geq p \geq N$.

Problem 4.

(a) Let $\sum_{n=1}^{\infty} a_n^{(\lambda)}$ $(\lambda \in \mathbb{N})$ be a collection of series which is *uniformly Cauchy* with respect to $\lambda \in \mathbb{N}$. Suppose that for every fixed $n \in \mathbb{N}$ the limit $\lim_{\lambda\to\infty} a_n^{(\lambda)} = a_n$ exists. Prove that $\sum_{n=1}^{\infty} a_n$ is Cauchy and

$$\lim_{\lambda\to\infty} \sum_{n=1}^{\infty} a_n^{(\lambda)} = \sum_{n=1}^{\infty} a_n \; .$$

(b) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with $a_n \in \mathbb{C}$ for $n \in \mathbb{N}$. Let M be a positive number and let Λ be any index set. For each $\lambda \in \Lambda$ let $b_n^{(\lambda)}$ $(n \in \mathbb{N})$ be a monotone sequence of real numbers bounded in absolute value by M. Prove that the collection of series $\sum_{n=1}^{\infty} b_n^{(\lambda)} a_n$ is uniformly Cauchy with respect to $\lambda \in \Lambda$. (*Hint:* use the discrete analogue of integration by parts.)

(c) (Abel's theorem) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with $a_n \in \mathbb{C}$ for $n \in \mathbb{N}$. Let $0 \leq x_\lambda < 1$ $(\lambda \in \mathbb{N})$ be a sequence whose limit is 1 as $\lambda \to \infty$. Show that

$$\lim_{\lambda\to\infty} \sum_{n=1}^{\infty} a_n \, (x_\lambda)^n = \sum_{n=1}^{\infty} a_n \; .$$

(*Hint:* apply (b) to the case $b_n^{(\lambda)} = (x_\lambda)^n$ and use (a).)

(d) (Cauchy product) Let $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$, and $\sum_{n=0}^{\infty} c_n = C$ be three convergent series whose terms are complex numbers. Assume that $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Show that $AB = C$. (*Hint:* let $x_\lambda = 1 - \frac{1}{\lambda}$ for $\lambda \in \mathbb{N}$ and apply (c) to

$$\sum_{n=0}^{\infty} c_n \, (x_\lambda)^n = \left(\sum_{n=0}^{\infty} a_n \, (x_\lambda)^n\right) \left(\sum_{n=0}^{\infty} b_n \, (x_\lambda)^n\right)$$

and let $\lambda \to \infty$.)

Problem 5. (Tauber's theorem – partial converse of Abel's theorem)

(a) Let $a_n \in \mathbb{C}$ for $n \in \mathbb{N}$. Assume that $\lim_{n\to\infty}(na_n) = 0$. Let $x_k \in \mathbb{C}$ with $|x_k| < 1$ for $k \in \mathbb{N}$ and $\lim_{k\to\infty} x_k = 1$. Suppose $f_k = \sum_{n=1}^{\infty} a_n (x_k)^n$ exists for $k \in \mathbb{N}$. Further suppose that $\lim_{k\to\infty} f_k = L$. Show that $\sum_{n=1}^{\infty} a_n = L$. (*Hint:* let N_k be the largest integer not exceeding $\frac{1}{1-|x_k|}$ and show

$$f_k - \sum_{n=1}^{N_k} a_n = \sum_{n=N_k+1}^{\infty} a_n (x_k)^n - \sum_{n=1}^{N_k} a_n (1-(x_k)^n)$$

approaches 0 as $k \to \infty$ by using Problem 3(a).)

(b) Use the example $a_n = (-1)^n$ to show that the condition $\lim_{n\to\infty}(na_n) = 0$ cannot be removed for the conclusion in (a) to hold.

Problem 6. (Double summation)

(a) Let $a_{m,n}$ be nonnegative real numbers for $m, n \in \mathbb{N}$. Show that if one of the following three series converges:

(i) $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} a_{m,n})$,
(ii) $\sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} a_{m,n})$,
(iii) $\sum_{N=2}^{\infty}\left(\sum_{m+n\le N} a_{m,n}\right)$,

then the other two converges and all three have the same limit.

(b) Show that for $|z| < 1$, one has

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \cdots + \frac{z^{2^n}}{1-z^{2^{n+1}}} + \cdots = \frac{z}{1-z}. \tag{1}$$

$$\frac{z}{1+z} + \frac{2z^2}{1+z^2} + \cdots + \frac{2^n z^{2^n}}{1-z^{2^n}} + \cdots = \frac{z}{1-z}. \tag{2}$$

(*Hint:* use (a) and the geometric series.)

Problem 7. A subset of $\mathbb{N}$ is said to be in *arithmetic progression* if it is of the form
$$\{a,\, a+d,\, a+2d,\, a+3d,\, \cdots\},$$
where $a,\, d \in \mathbb{N}$ and d is called the *step*. Show that $\mathbb{N}$ cannot be the disjoint union of a finite number of subsets that are in arithmetic progression with distinct steps except for the trivial case $a = d = 1$). (*Hint:* write $\sum_{n\in\mathbb{N}} z^n$ as a sum of terms of the type $\frac{z^a}{1-z^d}$.)

Stirling's Formula. In the following problem we assume as given the following formula of Stirling.
$$n! = \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \left(1 + E_n\right),$$
where $\lim_{n\to\infty} \frac{E_n}{n} = 0$.

Problem 8. Determine the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ when:

(a)
$$a_n = \frac{\sqrt[5]{n^2+3n+2} - \sqrt[5]{n^2+n+1}}{n^6}.$$

(b) $a_n = (\log n)^2$ (where log is the inverse function of the exponential function $x \mapsto e^x$).

(c) $a_n = n!$.

(d) $a_n = \frac{n^2}{4^n+3n}$.

(e) $a_n = \frac{(n!)^3}{(3n)!}$.

(f) Find the radius of convergence of the hypergeometric series of Gauss
$$F(\alpha,\beta,\gamma;z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\,\gamma(\gamma+1)\cdots(\gamma+n-1)}\, z^n,$$
where $\alpha,\, \beta,\, \gamma \in \mathbb{C}$ and $-\gamma+1 \notin \mathbb{N}$.

(g) Find the radius of the Bessel function of order r :
$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+r)!} \left(\frac{z}{2}\right)^{2n},$$
where r is a positive integer.