Math 55a, Assignment #5, October 17, 2003

Problem 1. (Implicit function theorem for convergent power series.) Let $r > 0$ and $a_{m,n} \in \mathbb{C}$ for $m, n \in \mathbb{N} \cup \{0\}$ with $|a_{m,n}| \leq \frac{1}{r^{m+n}}$ and $a_{0,0} = 0$. Let $F(z, w) = \sum_{k=1}^{\infty} \sum_{m+n=k} a_{m,n} z^m w^n$ for (z, w) satisfying $|z| < r$ and $|w| < r$. Assume $a_{0,1} \neq 0$.

(a) Prove that there exist $s > 0$ and a power series $\sum_{n=1}^{\infty} b_n z^n$ with complex coefficients which converges for $|z| < s$ such that $F(z, g(z)) = 0$ on $|z| < \gamma$ for some $\gamma > 0$, where $g(z) = \sum_{n=1}^{\infty} b_n z^n$.

(b) When $a_{m,n}$ is real for $m, n \in \mathbb{N} \cup \{0\}$, show that all b_n (for $n \in \mathbb{N}$) in Part (a) can be chosen to be real.

(c) When $a_{m,n}$ is real for $m, n \in \mathbb{N} \cup \{0\}$ and b_n from Part (a) is chosen to be real for $n \in \mathbb{N}$, show that if $0 < \delta < r$ and if $h(x)$ is a real-valued continuous function on $(-\delta, \delta)$ with $|h(x)| < r$ and $h(0) = 0$ such that $F(x, h(x)) = 0$ on $(-\delta, \delta)$, then $h(x) = g(x)$ on $(-\delta', \delta')$ for some $\delta' > 0$, where the function g is from Part (a). (*Hint:* if $\phi(x) = \sum_{n=1}^{\infty} c_n x^n$ with $c_n \in \mathbb{R}$ so that $c_1 > 0$ and $|c_n| \leq C\eta^n$ for some $\eta > 0$, from the identity $a^n - b^n = (a - b) \sum_{j=0}^{n-1} a^j b^{n-1-j}$ it follows that there exists some $\rho > 0$ which depends only on c_1, C, and η such that $\phi(x)$ is strictly increasing on $(-\rho, \rho)$.)

Justify carefully each step where the convergence of infinite series is used.

Problem 2. (Domain of convergence of a double series) Let $a_{m,n} \in \mathbb{C}$ for $m, n \in \mathbb{N} \cup \{0\}$. Let Ω be the subset of $(\mathbb{C} - \{0\}) \times (\mathbb{C} - \{0\})$ which is defined as follows. A point $(z_0, w_0) \in (\mathbb{C} - \{0\}) \times (\mathbb{C} - \{0\})$ belongs to Ω if and only if there exists a positive number r which may depend on (z_0, w_0) such that the series $\sum_{k=0}^{\infty} \sum_{m+n=k} a_{m,n} z^m w^n$ converges at (z, w) whenever $|z - z_0| < r$ and $|w - w_0| < r$. Assume that Ω is nonempty. Consider the map Φ from $(\mathbb{C} - \{0\}) \times (\mathbb{C} - \{0\})$ to $\mathbb{R} \times \mathbb{R}$ which sends (z, w) to $(\log |z|, \log |w|)$.

(a) Show that there exists some open subset G of $\mathbb{R} \times \mathbb{R}$ such that $\Omega = \Phi^{-1}(G)$.

(b) Show that whenever $(\xi_0, \eta_0) \in G$, the closed lower-left quadrant

$$\left\{ (\xi, \eta) \in \mathbb{R} \times \mathbb{R} \,\middle|\, \xi \leq \xi_0, \eta \leq \eta_0 \right\}$$

with vertex (ξ_0, η_0) is contained in G.

(c) Show that G is convex in the sense that, if $(\xi_1, \eta_1) \in G$ and $(\xi_2, \eta_2) \in G$, then the line segment

$$\left\{ (\xi, \eta) \in \mathbb{R} \times \mathbb{R} \,\middle|\, (\xi, \eta) = \lambda (\xi_1, \eta_1) + (1 - \lambda) (\xi_2, \eta_2) \text{ for some } 0 \leq \lambda \leq 1 \right\}$$

joining (ξ_1, η_1) and (ξ_2, η_2) is contained in G.

Problem 3. Let (X, d_X) and (Y, d_Y) be metric spaces. Assume that (X, d_X) is compact. Let f be a map from X to Y. Let $\Gamma_f \subset X \times Y$ be the graph of f which is defined as consisting of all points $(x, y) \in X \times Y$ with $y = f(x)$. Show that $f : X \to Y$ is continuous if and only if its graph Γ_f is a compact subset of $X \times Y$ when $X \times Y$ is given the metric

$$d_{X \times Y} \left((x_1, y_1), (x_2, y_2) \right) = d_X (x_1, x_2) + d_Y (y_1, y_2).$$

Problem 4. Let E be a dense subset of a metric space (X, d_X). Let (Y, d_Y) be a complete metric space in the sense that every Cauchy sequence in Y admits a limit in Y. Let $f : E \to Y$ be a map which is uniformly continuous in the sense that given any $\varepsilon > 0$ there exists some $\delta > 0$ such that $d_Y (a, b) < \varepsilon$ whenever $a, b \in E$ with $d_X (a, b) < \delta$. Show that there exists some continuous map $g : X \to Y$ such that the restriction of g to E agrees with f. (*Hint:* For each $p \in X$ and each positive integer n let $V_n(p)$ be the set of all $q \in E$ with $d_X (p, q) < \frac{1}{n}$. Show that the intersection of the closure of $f (V_n (p))$ for $n \in \mathbb{N}$ consists of a single point which we define as $g(p)$.)

Problem 5. Let (X, d_X) and (Y, d_Y) be compact metric spaces. For $k \in \mathbb{N}$ let $f_k : X \to Y$ be a map. Assume that the collection $\{f_k\}_{k \in \mathbb{N}}$ of maps is *equicontinuous* in the sense that for every $a \in X$ and every $\varepsilon > 0$ there exists some $\delta > 0$ (which may depend on a and ε) such that $d_Y (f_k(x), f_k(a)) < \varepsilon$ for all x satisfying $d_X (x, a) < \delta$ and for all $k \in \mathbb{N}$. Show that there exists $k_j \in \mathbb{N}$ for $j \in \mathbb{N}$ with $k_j < k_{j+1}$ such that the sequence $\left\{ f_{k_j} \right\}_{j \in \mathbb{N}}$ of maps converges uniformly on X to some continuous map $f : X \to Y$ in the sense that given any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $d_Y \left(f_{k_j}(x), f(x) \right) < \varepsilon$ for $j \geq N$ and $x \in X$. (*Hint:* first show that there exist some countable dense subset E of X and some increasing sequence of positive integers $\{k_j\}_{j \in \mathbb{N}}$ such that for every $a \in E$ the sequence $\left\{ f_{k_j}(a) \right\}_{j \in \mathbb{N}}$ of points in Y is a Cauchy sequence.)

Problem 6. (Problem 18 on Page 100 of Rudin's book) Every rational x can be written in the form $x = \frac{m}{n}$, where $n > 0$ and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{for} \quad x \text{ rational} \,, \\ \\ \frac{1}{n} & \text{for} \quad x = \frac{m}{n} \,. \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Problem 7. (Problem 18 on Page 100 of Rudin's book) Suppose f is a real-valued function with domain $\mathbb{R}$ which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b. Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed. Prove that f is continuous. (*Hint:* If $x_n \to x_0$, but $f(x_n) > r > f(x_0)$ for some r and all n, then $f(t_n) = r$ for some t_n between x_0 and x_n; thus $t_n \to x_0$. Find a contradiction.)

Problem 8. (Problems 23 and 24 on Page 101 of Rudin's book) A real-valued function f defined in (a, b) is said to be *convex* if

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$.

(a) Prove that every convex function is continuous.

(b) Prove that every increasing convex function of a convex function is convex.

(c) If f is convex in (a, b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

(d) Assume that f is a continuous real-valued function defined in (a, b) such that

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

for all $x, y \in (a, b)$. Prove that f is convex.

Problem 9. (Problem 25 on Page 102 of Rudin's book) If $A \subset \mathbb{R}^k$ and $B \subset \mathbb{R}^k$, define $A + B$ to be the set of all sums $\mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in A$ and $\mathbf{y} \in B$.

(a) If K is compact and C is closed in $\mathbb{R}^k$, prove that $K + C$ is closed. (*Hint:* Take $\mathbf{z} \notin K + C$. Put $F = \mathbf{z} - C$, the set of all $\mathbf{z} - \mathbf{y}$ with $\mathbf{y} \in C$. Then K and F are disjoint. Choose $\delta > 0$ such that $d(\mathbf{p}, \mathbf{q}) > \delta$ if $\mathbf{p} \in K$ and $\mathbf{q} \in F$. Show that the open ball with center $\mathbf{z}$ and radius δ does not intersect $K + C$.

(b) Let α be an irrational real number. Let C_1 be the set of all integers, let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of $\mathbb{R}$ whose sum $C_1 + C_2$ is not closed, by showing that $C_1 + C_2$ is a countable dense subset of $\mathbb{R}$.

Problem 10. Let X be a compact metric space and let f_n be a continuous map from X to $\mathbb{R}$ for $n \in \mathbb{N}$. Let f be a continuous map from X to $\mathbb{R}$. Assume that for each $x \in X$ the sequence $\{f_n(x)\}_{n \in \mathbb{N}}$ in $\mathbb{R}$ is monotonically nondecreasing and converges to $f(x)$. Show that f_n converges to f uniformly on X in the sense that, given any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and for all $x \in X$.