Math 55a, Assignment #6, October 24, 2003

Problem 1. (Problem 8 on Page 35 and Problem 9 on Page 59 of Rudin's book)

(a) Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \;\middle|\; x_1 = 3x_2 \text{ and } x_3 = 7x_4 \right\}.$$

Find a basis of U (over $\mathbb{R}$).

(b) Prove that if T is a linear map from $\mathbb{R}^4$ to $\mathbb{R}^2$ such that

$$\text{null } T = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; x_1 = 5x_2 \text{ and } x_3 = 7x_4 \right\},$$

then T is surjective.

Problem 2. (Quotient vector spaces) Let V be a vector space over a field $\mathbb{F}$ and W be an $\mathbb{F}$-vector subspace of V. Let $\sim$ be the equivalence relation on V defined as follows. Two elements v_1 and v_2 of V are equivalent (in notations, $v_1 \sim v_2$) if and only if $v_1 - v_2$ belongs to W. Let Q be the set of equivalence classes (in notations, $Q = V/\sim$). Let $[v]$ be the equivalence class in V containing the element v of V (*i.e.,* $[v]$ is the set of all elements of V which are equivalent to v). Define addition in Q by $[v_1] + [v_2] = [v_1 + v_2]$ and scalar multiplication by $a[v] = [av]$ for $a \in \mathbb{F}$ and v, v_1, $v_2 \in V$.

(a) Show that the above procedure yields a vector space Q over $\mathbb{F}$ (which is called the *quotient vector space* of V by the subspace W).

(b) Define the map $T : V \to Q$ by $T(v) = [v]$. Show that T is a linear map from V to Q over $\mathbb{F}$ (in notations, $T \in \mathcal{L}(V, Q)$ or more precisely $T \in \mathcal{L}_{\mathbb{F}}(V, Q)$). ($T$ is the called the *projection* onto the quotient space.)

(c) Show that the range of T is Q and the null space of T is precisely W.

(d) Let U be an $\mathbb{F}$-vector subspace of V. Show that $T^{-1}(T(U)) = U + W$. By applying to $T|_U$ and T_{U+W} the formula that the dimension of the domain vector space is equal to the sum of the dimension of the null space and that of the range of a linear map, verify the formula that

$$\dim_{\mathbb{F}} U + \dim_{\mathbb{F}} W = \dim_{\mathbb{F}} (U \cap W) + \dim_{\mathbb{F}} (U + W).$$

Problem 3. (Extension of the field of definition of a vector space from $\mathbb{R}$ to $\mathbb{C}$) Let V be a vector space over the real number field $\mathbb{R}$. Show that V can be made into a vector space over $\mathbb{C}$ (with scalar multiplication by real numbers compatible with the given $\mathbb{R}$-vector space structure of V) if and only if there exists an $\mathbb{R}$-linear map T from V to itself such that $T^2 = -\mathrm{id}_V$, where T^2 means $T \circ T$ and id_V means the identity map of V. In such a case, show that $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V$. (*Hint:* T is given by the scalar multiplication by $\sqrt{-1}$.)

Problem 4. (More general extension of fields of definition of vector spaces) Let $P(x) = \sum_{j=0}^{n} a_j x^j$ be an *irreducible* polynomial of degree $n > 1$ whose coefficients are rational numbers. Let $\theta \in \mathbb{C}$ be a root of $P(x)$. Let $\mathbb{F}$ be the set of all complex numbers of the form $\frac{R(\theta)}{Q(\theta)}$, where $R(x)$ and $Q(x)$ are any polynomials with coefficients in $\mathbb{Q}$ and $Q(\theta) \neq 0$.

(a) Show that $\mathbb{F}$ is a field when its addition and multiplication are inherited from $\mathbb{C}$ (in other words, $\mathbb{F}$ is a subfield of $\mathbb{C}$).

(b) When scalar multiplication is defined as ordinary multiplication in $\mathbb{C}$, show that $\mathbb{F}$ is a vector space over $\mathbb{Q}$ and its dimension $\dim_{\mathbb{Q}} \mathbb{F}$ is equal to n.

(c) Let V be a vector space over $\mathbb{Q}$. Show that V can be made into a vector space over $\mathbb{F}$ (with scalar multiplication by rational numbers compatible with the given $\mathbb{Q}$-vector space structure of V) if and only if there exists a $\mathbb{Q}$-linear map T from V to itself such that $\sum_{j=0}^{n} a_j T^j = 0$ in $\mathcal{L}_{\mathbb{Q}}(V, V)$, where T^j means the composite map formed from j copies of T and $\mathcal{L}_{\mathbb{Q}}(V, V)$ means the algebra of all $\mathbb{Q}$-linear maps from V to itself. In such a case, show that $\dim_{\mathbb{Q}} V = n \dim_{\mathbb{F}} V$.

Problem 5. (Dual vector spaces and tensor products of vector spaces) Let $\mathbb{F}$ be a field and V be an $\mathbb{F}$-vector space. The *dual vector space* of V is defined as the space $\mathcal{L}_{\mathbb{F}}(V, \mathbb{F})$ of all $\mathbb{F}$-linear maps from V to $\mathbb{F}$ when $\mathbb{F}$ is regarded as an $\mathbb{F}$-vector space of dimension 1. Let W be an $\mathbb{F}$-vector space. Let

$$T : V \times W \to \mathcal{L}_{\mathbb{F}}\left(\mathcal{L}_{\mathbb{F}}(V, \mathbb{F}), W\right)$$

be defined by

$$(T(v, w))(f) = f(v) \cdot w \quad \text{for } v \in V, w \in W, \text{ and } f \in \mathcal{L}_{\mathbb{F}}(V, \mathbb{F}).$$

Denote $T(v, w)$ by $v \otimes w$.

(a) Show that $v \otimes (aw) = (av) \otimes w = a\,(u \otimes w)$ for $v \in V$, $w \in W$, and $a \in \mathbb{F}$.

(b) *(Basis of tensor product)* Let $v_1, \cdots, v_m$ be an $\mathbb{F}$-basis of V and $w_1, \cdots, w_n$ be an $\mathbb{F}$-basis of W. Show that the set

$$\left\{ v_i \otimes w_j \;\middle|\; 1 \leq i \leq m, 1 \leq j \leq n \right\}$$

of mn elements of $\mathcal{L}_{\mathbb{F}}\left(\mathcal{L}_{\mathbb{F}}\left(V, \mathbb{F}\right), W\right)$ forms an $\mathbb{F}$-basis of $\mathcal{L}_{\mathbb{F}}\left(\mathcal{L}_{\mathbb{F}}\left(V, \mathbb{F}\right), W\right)$. (The $\mathbb{F}$-vector space $\mathcal{L}_{\mathbb{F}}\left(\mathcal{L}_{\mathbb{F}}\left(V, \mathbb{F}\right), W\right)$ is called the *tensor product* of V and W and is denoted by $V \otimes W$ or more precisely by $V \otimes_{\mathbb{F}} W$. This is one of a number of equivalent ways to define the tensor product of two vector spaces.)

(c) *(Alternative definition of tensor product)* Let $\mathcal{B}il_{\mathbb{F}}\left(V, W, \mathbb{F}\right)$ denote the set of all $\mathbb{F}$-bilinear maps f from $V \times W$ to $\mathbb{F}$ (*i.e.,* for any fixed $w \in W$ the map $v \mapsto f(v, w)$ belongs to $\mathcal{L}_{\mathbb{F}}(V, \mathbb{F})$ and for any fixed $v \in V$ the map $w \mapsto f(v, w)$ belongs to $\mathcal{L}_{\mathbb{F}}(W, \mathbb{F})$). The set $\mathcal{B}il_{\mathbb{F}}\left(V, W, \mathbb{F}\right)$ is a $\mathbb{F}$-vector space with the usual operations of addition of $\mathbb{F}$-valued functions and and multiplication of an $\mathbb{F}$-valued function by a scalar. Show that there exists a unique $\mathbb{F}$-linear map $\Xi : V \otimes W \to \mathcal{L}_{\mathbb{F}}\left(\mathcal{B}il_{\mathbb{F}}\left(V, W, \mathbb{F}\right), \mathbb{F}\right)$ such that $\Xi(v \otimes w) = f(v, w)$ for $f \in \mathcal{B}il_{\mathbb{F}}\left(V, W, \mathbb{F}\right)$. Verify that Ξ is bijective.

(d) *(Linear maps between tensor products)* Let V' and W' be $\mathbb{F}$-vector spaces and $R \in \mathcal{L}_{\mathbb{F}}\left(V, V'\right)$ and $S \in \mathcal{L}_{\mathbb{F}}\left(W, W'\right)$. Define a map $R \otimes_{\mathbb{F}} S$ from $V \otimes W$ to $V' \otimes W'$ as follows, when we use the definitions $V \otimes W = \mathcal{L}_{\mathbb{F}}\left(\mathcal{L}_{\mathbb{F}}\left(V, \mathbb{F}\right), W\right)$ and $V' \otimes W' = \mathcal{L}_{\mathbb{F}}\left(\mathcal{L}_{\mathbb{F}}\left(V', \mathbb{F}\right), W'\right)$. For $f \in \mathcal{L}_{\mathbb{F}}\left(V, \mathbb{F}\right)$ and $f' \in \mathcal{L}_{\mathbb{F}}\left(V', \mathbb{F}\right)$ and $v' \in V'$,

$$\left(\left(R \otimes_{\mathbb{F}} S\right)(v, w)\right)\left(f'\right) = f'\left(Rv\right) S(w).$$

Show that $R \otimes_{\mathbb{F}} S$ is an $\mathbb{F}$-linear map from $V \otimes W$ to $V' \otimes W'$.

(e) *(Extension of field of definition)* Let $\mathbb{F}$ be a subfield of a field $\mathbb{K}$. If the $\mathbb{F}$-vector space W is also a $\mathbb{K}$-vector space, show that the scalar multiplication defined by $(a, T) \mapsto aT$, in the sense that $(aT)\,(f) = af$ for $a \in \mathbb{K}$ and $f \in \mathcal{L}_{\mathbb{F}}\left(\mathcal{L}_{\mathbb{F}}\left(V, \mathbb{F}\right), W\right)$, makes $V \times_{\mathbb{F}} W$ a $\mathbb{K}$-vector space. Verify that the map from V to $V \otimes_{\mathbb{F}} \mathbb{K}$ defined by $v \mapsto v \otimes 1$ for $v \in \mathbb{V}$ is injective and is $\mathbb{F}$-linear, where 1 is the multiplicative identity element of $\mathbb{K}$. (We use this map to identify V as an $\mathbb{F}$-vector subspace of $V \otimes_{\mathbb{F}} \mathbb{K}$.)

(f) *(Almost complex structure)* Let U be a vector space over $\mathbb{R}$. Let ρ the $\mathbb{R}$-linear map from $U \otimes_{\mathbb{R}} \mathbb{C}$ to itself which sends $u \otimes \sqrt{-1}$ to $-\left(u \otimes \sqrt{-1}\right)$ for $u \in U$. (We call ρ the conjugation map.) Show that U can be made into a vector space over $\mathbb{C}$ (with scalar multiplication by real numbers compatible with the given $\mathbb{R}$-vector space structure of U) if and only if there exists a $\mathbb{C}$-vector subspace W of the $\mathbb{C}$-vector space $U \otimes_{\mathbb{R}} \mathbb{C}$ such that $\mathbb{C}$-vector space $U \otimes_{\mathbb{R}} \mathbb{C}$ is equal to $W + \rho(W)$ and $W \cap \rho(W) = \{0\}$. (*Hint:* For the "if" part, consider the projection map $\pi : U \otimes_{\mathbb{R}} \mathbb{C} \to W$ defined by $(w + w') \mapsto w$ for $w \in W$ and $w' \in \rho(W)$ and verify that $\pi|_U : U \to W$ is bijective and $\mathbb{R}$-linear when U is naturally regarded as an $\mathbb{R}$-vector subspace of $U \otimes_{\mathbb{R}} \mathbb{C}$ according to (e). For the "only if" part, let $J : U \otimes_{\mathbb{R}} \mathbb{C} \to U \otimes_{\mathbb{R}} \mathbb{C}$ be the $\mathbb{C}$-linear map defined by $u \otimes c \mapsto \left(\sqrt{-1}\, u\right) \otimes c$ for $u \in U$ and $c \in \mathbb{C}$ and set T to be the null space of $J - \sqrt{-1}\,\mathrm{id}$, where id is the identity map of $U \otimes_{\mathbb{R}} \mathbb{C}$.) *Terminology:* Given an $\mathbb{R}$-vector space U, a decomposition $U_{\mathbb{R}} \otimes \mathbb{C} = T \oplus \bar{T}$ is called an *almost complex structure* of U where T is a $\mathbb{C}$-vector subspace of $U_{\mathbb{R}} \otimes \mathbb{C}$ and $\bar{T}$ means the image $\rho(T)$ of T under the conjugation map ρ.

Problem 6. Let $\mathbb{F}$ be a field and for $1 \le k \le n$ let V_k be a finite dimensional $\mathbb{F}$-vector space. Let $T_k : V_k \to V_{k+1}$ be an $\mathbb{F}$-linear map for $1 \le k \le n-1$ such that $T_{k+1} \circ T_k = 0$ for $1 \le k \le n-2$. Let R_k be the range of T_k and K_k be the null space of T_k for $1 \le k \le n-1$. Let $K_n = V_n$ and $R_0 = \{0\}$. Let H_k be the quotient vector space K_k / R_{k-1} for $1 \le k \le n$. Prove that

$$\sum_{k=1}^{n} (-1)^k \dim_{\mathbb{F}} H_k = \sum_{k=1}^{n} (-1)^k \dim_{\mathbb{F}} V_k.$$

Problem 7. (Five-lemma) Let $\mathbb{F}$ be a field. In the diagram below, V_j and W_j are $\mathbb{F}$-vector spaces and θ_j and φ_k and ψ_k are $\mathbb{F}$-linear maps for $1 \le j \le 5$ and $1 \le k \le 4$.

$$\begin{array}{ccccccccc}
V_1 & \xrightarrow{\varphi_1} & V_2 & \xrightarrow{\varphi_2} & V_3 & \xrightarrow{\varphi_3} & V_4 & \xrightarrow{\varphi_4} & V_5 \\
\theta_1 \downarrow & & \theta_2 \downarrow & & \theta_3 \downarrow & & \theta_4 \downarrow & & \theta_5 \downarrow \\
W_1 & \xrightarrow{\psi_1} & W_2 & \xrightarrow{\psi_2} & W_3 & \xrightarrow{\psi_3} & W_4 & \xrightarrow{\psi_4} & W_5
\end{array}$$

Assume that both rows in the diagram are exact in the sense that

$$\begin{cases} \text{null } \varphi_{j+1} = \text{range } \varphi_j \\ \\ \text{null } \psi_{j+1} = \text{range } \psi_j \end{cases}$$

for $1 \leq j \leq 3$. Assume that the diagram is commutative in the sense that $\psi_j \circ \theta_j = \theta_{j+1} \circ \varphi_j$ for $1 \leq j \leq 4$. Show that, if θ_1, θ_2, θ_4, θ_5 are all bijective, then θ_3 is also bijective.

Problem 8. (Generator of an ideal of polynomials of one variable) Let $\mathbb{F}$ be a field and $P_1(x), \cdots, P_k(x)$ be a finite number of (non identically zero) polynomials of positive degree in a single variable x with coefficients in $\mathbb{F}$. Let $\mathcal{I}$ be the set of all polynomials of the form $\sum_{j=1}^{k} Q_j(x)P_j(x)$, where $Q_1(x), \cdots, Q_k(x)$ are polynomials in x with coefficients in $\mathbb{F}$. Let $R(x)$ be an element of $\mathcal{I}$ so that the degree of $R(x)$ is the minimum among all elements of $\mathcal{I}$. Show that any element $P(x)$ of $\mathcal{I}$ can be written as $P(x) = Q(x)R(x)$ for some polynomial $Q(x)$ in x with coefficients in $\mathbb{F}$. *Terminology:* $\mathcal{I}$ is an *ideal* of the ring of polynomials of one variable with coefficients in $\mathbb{F}$ and $R(x)$ is a generator of $\mathcal{I}$.

Problem 9. (A proof of the fundamental theorem of algebra)

(a) Let $Q(w) = 1 + \sum_{n=1}^{\infty} \alpha_n w^n$ be a power series with complex coefficients and a positive radius of convergence. Let k be any positive integer. Show that there exists a unique power series $s(w) = 1 + \sum_{n=1}^{\infty} a_n w^n$ with a radius of convergence $\geq R > 0$ such that $(s(w))^k = Q(w)$ for $|w| < R$.

(b) Let $c \in \mathbb{C}$ and $P(z)$ be a polynomial in a single variable w with complex coefficients such that with $P(c) \neq 0$. Show that there exist a nonzero complex number A, a positive integer k, and a power series $t(z) = 1+\sum_{n=1}^{\infty} b_n(z-c)^n$ with complex coefficients and a radius of convergence $\geq R > 0$ such that

$$(z-c)^k \left(t(z)\right)^k = A\left(\frac{1}{P(z)} - \frac{1}{P(c)}\right)$$

for $|z| < R$. (*Hint:* write $P(z) - P(c) = (z-c)^k P_1(z)$ with $P_1(c) \neq 0$ and

$$A\left(\frac{1}{P(z)} - \frac{1}{P(c)}\right) = (z-c)^k \left(1 + \sum_{k=1}^{\infty} \alpha_n (z-c)^n\right)$$

for some nonzero complex number A and apply (a).)

(c) Let $P(z)$ be a polynomial of positive degree in one variable z with complex coefficients. Show that $P(z)$ admits at least one root. (*Hint:* Assume the contrary. Then

$$\left|\frac{1}{P(c)}\right| = \sup_{z\in\mathbb{C}}\left|\frac{1}{P(z)}\right|$$

for some $c \in \mathbb{C}$. By the implicit function theorem for power series (Assignment#5, Problem 1) applied to $w = (z-c)t(z)$, we can find a power series $\sigma(w) = c + \sum_{n=1}^{\infty}\beta_n w^n$ such that $w = (\sigma(w)-c)\,t(\sigma(w))$. Then

$$w^k = A\left(\frac{1}{P(\sigma(w))} - \frac{1}{P(c)}\right),$$

which means that the image of $\frac{1}{P(z)} - \frac{1}{P(c)}$ for z in some neighborhood of c covers some neighborhood of the origin, contradicting that

$$\left|\frac{1}{P(z)}\right| \le \left|\frac{1}{P(c)}\right|$$

for z in any neighborhood of c.)