Math 55a, Assignment #7, October 31, 2003

Definitions and Notations. Let V be a finite-dimensional vector space over a field $\mathbb{F}$. The set of all $\mathbb{F}$-linear maps from V to itself is denoted by $\mathrm{End}_{\mathbb{F}}(V)$.

An $\mathbb{F}$-vector subspace $\mathcal{L}$ of $\mathrm{End}_{\mathbb{F}}(V)$ is called a *Lie algebra* (or more precisely, a Lie subalgebra of $\mathrm{End}_F(V)$) if $ST - TS$ belongs to $\mathcal{L}$ whenever both S and T belong to $\mathcal{L}$.

One denotes $ST - TS$ by $[S, T]$. For subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{L}$ let $[\mathcal{A}, \mathcal{B}]$ be the set of all $[A, B]$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

For $A \in \mathcal{L}$, one uses $\mathrm{ad}_{\mathcal{L}} A$ (or simply $\mathrm{ad}\, A$) to denote the element of $\mathrm{End}_{\mathbb{F}}(\mathcal{L})$ which sends $S \in \mathcal{L}$ to $[A, S]$. For a subset $\mathcal{A}$ of $\mathcal{L}$ one uses $\mathrm{ad}_{\mathcal{L}}\mathcal{A}$ (or simply $\mathrm{ad}\,\mathcal{A}$) to denote the set of all $\mathrm{ad}_{\mathcal{L}} A$ with $A \in \mathcal{A}$.

Inductively let $\mathcal{L}^1 = \mathcal{L}^{(1)} = \mathcal{L}$, $\mathcal{L}^n = [\mathcal{L}, \mathcal{L}^{n-1}]$, and $\mathcal{L}^{(n)} = \left[\mathcal{L}^{(n-1)}, \mathcal{L}^{(n-1)}\right]$ for integers $n \geq 2$. The Lie algebra $\mathcal{L}$ is called *nilpotent* if $\mathcal{L}^N = 0$ for some positive integer N. The Lie algebra $\mathcal{L}$ is called *solvable* if $\mathcal{L}^{(N)} = 0$ for some positive integer N.

An $\mathbb{F}$-vector subspace $\mathcal{I}$ of $\mathcal{L}$ is called an *ideal* if $[S, T] \in \mathcal{I}$ for all $S \in \mathcal{L}$ and $T \in \mathcal{I}$.

Problems 1-7 below (with $\mathbb{F} = \mathbb{C}$) lead to the construction of an $\mathbb{F}$-basis of V with respect to which the matrix of every element in certain subsets of $\mathcal{L}_{\mathbb{F}}(V)$ can be simultaneously put into upper triangular form. The subsets being considered are solvable Lie algebras.

Consider the following statement which is indexed by a positive integer n and is meant for all V and $\mathcal{L}$ as long as the dimension of $\mathcal{L}$ over $\mathbb{C}$ as a $\mathbb{C}$-vector space is n.

$(\dagger)_n$ Let V be a $\mathbb{C}$-vector space of finite positive dimension. Let $\mathcal{L}$ be a Lie subalgebra of $\mathrm{End}_{\mathbb{C}}(V)$. Let n be the dimension of $\mathcal{L}$ over $\mathbb{C}$ as a $\mathbb{C}$-vector space. If for every $S \in \mathcal{L}$ there exists some positive integer $N = N_S$ (which may depend on S) such that $S^N = 0$ as an element of $\mathrm{End}_{\mathbb{C}}(V)$, then there exists some nonzero element $v \in V$ such that $Sv = 0$ for all $S \in \mathcal{L}$.

Problem 1. Let V be a $\mathbb{C}$-vector space of finite positive dimension. Let $\mathcal{L}$ be a Lie subalgebra of $\mathrm{End}_{\mathbb{C}}(V)$. Let n be the dimension of $\mathcal{L}$ over $\mathbb{C}$ as a $\mathbb{C}$-vector space. Let $\mathcal{K}$ be a maximal *proper* Lie subalgebra of $\mathcal{L}$ (which means that $\mathcal{K}$ is a Lie subalgebra of $\mathcal{L}$ not equal to $\mathcal{L}$ and that, if $\mathcal{A}$ is a Lie subalgebra of $\mathcal{L}$ which contains $\mathcal{K}$ but not equal to $\mathcal{K}$, then $\mathcal{A}$ must be equal to $\mathcal{L}$). *Assume that Statement* $(\dagger)_k$ *holds for any* $k \leq n-1$. Suppose that for every $S \in \mathcal{K}$ there exists some positive integer $N = N_S$ (which may depend on S) such that $S^N = 0$ as an element of $\mathrm{End}_{\mathbb{C}}(V)$. Prove that $\mathcal{K}$ is an ideal of $\mathcal{L}$. (*Hint:* Consider the subset $\mathrm{ad}_{\mathcal{L}}\mathcal{K}$ of $\mathrm{End}_{\mathbb{C}}(\mathcal{L})$ and let $\mathcal{T}$ be the subset of $\mathrm{End}_{\mathbb{C}}(\mathcal{L}/\mathcal{K})$ induced by $\mathrm{ad}_{\mathcal{L}}\mathcal{K}$. Let ℓ be the dimension of $\mathcal{T}$ as a $\mathbb{C}$-vector space. Apply $(\dagger)_\ell$ to the case where V (respectively $\mathcal{L}$) in it is replaced by $\mathcal{L}/\mathcal{K}$ (respectively $\mathcal{T}$) to find A in $\mathcal{L}$ but not in $\mathcal{K}$ such that $[S, A] \in \mathcal{K}$ for all $S \in \mathcal{K}$. Verify that $\mathcal{K} + \mathbb{C}A$ is a Lie subalgebra of $\mathcal{L}$. Need to check that $\ell \leq n-1$ and that for every $T \in \mathcal{T}$ there exists some integer m which may depend on T such that $T^m = 0$ as element of $\mathrm{End}_{\mathbb{C}}(\mathcal{L}/\mathcal{K})$.)

Problem 2. Use Problem 1 to prove $(\dagger)_n$ by induction on n. (*Hint:* Let $\mathcal{K}$ be a maximal Lie subalgebra of $\mathcal{L}$ (and allow $\mathcal{K}$ to be just $\{0\}$). By Problem 1, $\mathcal{K}$ is an ideal and $\dim_{\mathbb{C}} \mathcal{K}$ must be $n-1$. Let W be the $\mathbb{C}$-vector subspace of V consisting of all $v \in V$ with $Sv = 0$ for all $S \in \mathcal{K}$. By $(\dagger)_{n-1}$ the dimension of W over $\mathbb{C}$ must be positive. Pick some element S in $\mathcal{L}$ but not in $\mathcal{K}$ and, by considering the element of $\mathrm{End}_{\mathbb{C}}(W)$ induced by S, verify that the null space of S contains a nonzero element of W.)

Problem 3. Use Problem 2 to prove the following. Let V be a $\mathbb{C}$-vector space of finite positive dimension m. Let $\mathcal{L}$ be a Lie subalgebra of $\mathrm{End}_{\mathbb{C}}(V)$. If for every $S \in \mathcal{L}$ there exists some positive integer $N = N_S$ (which may depend on S) such that $S^N = 0$, then there exists a $\mathbb{C}$-basis $e_1, \cdots, e_m$ of V such that $Se_j \in \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_{j-1}$ for $1 \leq j \leq m$ and for any $S \in \mathcal{L}$ and, as a consequence, the matrix of every element of $\mathcal{L}$ with respect to the $\mathbb{C}$-basis $e_1, \cdots, e_m$ of V is in upper triangular form with all the diagonal entries zero.

Problem 4. Use Problem 3 to prove the following theorem of Engel. Let $\mathcal{L}$ be a Lie algebra (of finite positive dimension over $\mathbb{C}$). Show that there exists some positive integer p such that

$$[S_1, [S_2, \cdots, [S_{p-1}, S_p] \cdots]] = 0$$

for all $S_1, S_2, \cdots, S_p \in \mathcal{L}$ if and only if for every $S \in \mathcal{L}$ there exists some positive integer $N = N_S$ (which may depend on S) such that

$$[S_1, [S_2, \cdots, [S_N, T] \cdots]] = 0$$

for all $T \in \mathcal{L}$ when $S_j = S$ for $1 \leq j \leq N$. In other words, if for every $S \in \mathcal{L}$ one has $(\mathrm{ad}_{\mathcal{L}} S)^N = 0$ for some positive integer $N = N_S$ (which may depend on S), then $\mathcal{L}$ is a nilpotent Lie algebra. (*Hint:* Apply Problem 3 when V (respectively $\mathcal{L}$) is replaced by $\mathcal{L}$ and $\mathrm{ad}_{\mathcal{L}}\mathcal{L}$.)

Problem 5. Let $\mathbb{F}$ be a field and V be a $\mathbb{F}$-vector space of finite positive dimension n. Let $\mathcal{L}$ be a Lie subalgebra of $\mathrm{End}_{\mathbb{F}}(V)$. Assume that $\mathcal{L}$ is solvable as a Lie algebra. Show that there exists an ideal $\mathcal{K}$ in $\mathcal{L}$ which, as a $\mathbb{F}$-vector subspace of $\mathcal{L}$, has dimension $n-1$ over $\mathbb{F}$. (*Hint:* verify that $[\mathcal{L}, \mathcal{L}]$ is a *proper* $\mathbb{F}$-vector subspace of $\mathcal{L}$ and choose $\mathcal{K}$ to be a *proper* $\mathbb{F}$-vector subspace of $\mathcal{L}$ which contains $[\mathcal{L}, \mathcal{L}]$.)

Problem 6. Let $\mathbb{F}$ be a field such that the sum of m copies of the identity element 1 of F is a nonzero element of F for any positive integer m. Let V be an $\mathbb{F}$-vector space of finite positive dimension n. Let $\mathcal{L}$ be a Lie subalgebra of $\mathrm{End}_{\mathbb{F}}(V)$ and $\mathcal{K}$ be an ideal in $\mathcal{L}$.

(a) Let v be a nonzero element of V and λ be an $\mathbb{F}$-linear map from $\mathcal{K}$ to $\mathbb{F}$ such that $Sv = \lambda(S)v$ for $S \in \mathcal{K}$. Let $T \in \mathcal{L}$ and ℓ be the smallest integer such that $v, Tv, \cdots, T^\ell v$ are $\mathbb{F}$-linearly dependent. Let U be the ℓ-dimensional $\mathbb{F}$-vector space of V spanned by $v, Tv, \cdots, T^\ell v$. Show that every element $S \in \mathcal{K}$ maps U to U and its matrix with respect to the $\mathbb{F}$-basis $v, Tv, \cdots, T^{\ell-1}v$ of V is in upper triangular form whose diagonal entries are all equal to $\lambda(S)$. As a consequence, show that $\lambda([S,T]) = 0$ for every $S \in \mathcal{K}$. (*Hint:* For any $S \in \mathcal{K}$, use $ST^jv = TST^{j-1}v + [S,T]\,T^{j-1}v$ and $[S,T] \in \mathcal{K}$ and induction on j to show that ST^jv is equal to $\lambda(S)T^jv$ plus some element of the $\mathbb{F}$-vector subspace of V spanned by $v, Tv, \cdots, T^{j-1}v$.)

(b) Let λ be an $\mathbb{F}$-linear map from $\mathcal{K}$ to $\mathbb{F}$ and let W be the $\mathbb{F}$-vector subspace of V consisting of all $w \in V$ such that $Sw = \lambda(S)w$ for all $S \in \mathcal{K}$. Show that every element T of $\mathcal{L}$ maps W to W. (*Hint:* For $S \in \mathcal{K}$ use $STw = TSw + [S,T]\,w$ for all $S \in \mathcal{K}$ and use the last conclusion of Part (a).)

Problem 7. Let V be a $\mathbb{C}$-vector space of finite positive dimension n. Let $\mathcal{L}$ be a *solvable* Lie subalgebra of $\mathrm{End}_{\mathbb{C}}(V)$. Prove that there exists some nonzero element v of V such that v is an eigenvector for every element of $\mathcal{L}$. Prove, as a consequence, the following theorem of Lie. There exists a

$\mathbb{C}$-basis of V with respect to which the matrix of every element of $\mathcal{L}$ is in upper triangular form. (*Hint:* Use Problem 5 to find an ideal $\mathcal{K}$ in $\mathcal{L}$ which has dimension $n-1$ over $\mathbb{C}$. Let W be the $\mathbb{C}$-vector subspace of V consisting of all elements $w \in V$ (plus 0) which are at the same time eigenvectors of all elements of $\mathcal{K}$. By induction on $n-1$, W is nonzero. Pick $T \in \mathcal{L}$ which is not in $\mathcal{K}$. By Part (b) of Problem 6, T maps W to W. Choose an element of W which is an eigenvector of T.)