Math 55a, Assignment #8, November 7, 2003

Notations. $\mathbb{R}$ is the field of all real numbers. $\mathbb{C}$ is the field of all complex numbers. $\mathbb{N}$ denotes the set of all natural numbers (*i.e.,* all positive integers). For a field $\mathbb{F}$ and $\mathbb{F}$-vector spaces V and W, $\mathrm{Hom}_{\mathbb{F}}(V,W)$ denotes the set of all $\mathbb{F}$-linear maps from V to W and $\mathrm{End}_{\mathbb{F}}(V)$ denotes the set of all $\mathbb{F}$-linear maps from V to itself. For $T \in \mathrm{Hom}_{\mathbb{F}}(V,W)$, $\mathrm{Ker}\, T$ denotes the null space (*i.e.* the kernel) of T and $\mathrm{Im}\, T$ denotes the range (*i.e.* the image) of T. The identity map of A is denoted by id_A. The ring of all polynomials in a single variable X with coefficients in $\mathbb{F}$ is denoted by $\mathbb{F}[X]$.

Problem 1. (Problems 5 and 6 on Page 94 of Axler's book) Let $S \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$ be defined by $S(w,z) = (z,w)$ for $z, w \in \mathbb{C}$. Let $T \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^3)$ be defined by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$ for $z_1, z_2, z_3 \in \mathbb{C}$. Find all the eigenvalues and eigenvectors of S and T.

Problem 2. (Problem 4 on Page 158 of Axler's book) Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let V be an $\mathbb{F}$-vector space of positive finite dimension with an inner product. Suppose $P \in \mathrm{End}_{\mathbb{F}}(V)$ such that $P^2 = P$. Show that P is an orthogonal projection if and only if P is self-adjoint.

Problem 3. (Problem 14 on Page 159 of Axler's book) Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let V be an $\mathbb{F}$-vector space of positive finite dimension with an inner product. Suppose $T \in \mathrm{End}_{\mathbb{F}}(V)$ is self-adjoint. Let $\lambda \in \mathbb{F}$ and $\varepsilon > 0$. Prove that if there exists $v \in V$ such that $\|v\| = 1$ and $\|Tv - \lambda v\| < \varepsilon$, then T has an eigenvalue λ' such that $|\lambda - \lambda'| < \varepsilon$.

Problem 4. (Inner product of the underlying $\mathbb{R}$-vector space structure of a $\mathbb{C}$-vector space) Let V be an $\mathbb{R}$-vector space of finite positive dimension with an $\mathbb{R}$-basis $e_1, \cdots, e_n$ so that $V = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n$. Let $\tilde{V} = V \otimes_{\mathbb{R}} \mathbb{C}$ and we identify $\tilde{V}$ with $\mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$. For $v = \sum_{j=1}^n a_j e_j \in \tilde{V}$ use $\bar{v}$ to denote $\sum_{j=1}^n \bar{a}_j e_j$, where $\bar{a}_j$ is the complex-conjugate of a_j. For a subset A of $\tilde{V}$ let $\bar{A}$ denote the set of all $\bar{v}$ for $v \in A$.

Let $g(u,v)$ be an $\mathbb{R}$-bilinear function on $V \times V$ which defines an inner product of the $\mathbb{R}$-vector space V. Let $\tilde{g}(u,v)$ be the $\mathbb{C}$-bilinear function on $\tilde{V} \times \tilde{V}$ which is the extension of $g(u,v)$. In other words, $\tilde{g}(u,v)$ is $\mathbb{C}$-linear in $u \in \tilde{V}$ for fixed $v \in \tilde{V}$ and is $\mathbb{C}$-linear in $v \in \tilde{V}$ for fixed $u \in \tilde{V}$ and $\tilde{g}(u,v) = g(u,v)$ when both u and v are in the subset $V = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n$ of $\tilde{V} = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$.

Now assume that there exists some $J \in \operatorname{End}_{\mathbb{R}}(V)$ with $J^2 = -\mathrm{id}_V$ so that V can be regarded as a $\mathbb{C}$-vector space where multiplication of an element v of V by $\sqrt{-1}$ yields Jv. Let $\tilde{J} \in \operatorname{End}_{\mathbb{C}}\left(\tilde{V}\right)$ be the $\mathbb{C}$-linear extension of J. Denote the $\mathbb{C}$-vector subspace

$$\operatorname{Ker}\left(\tilde{J} - \sqrt{-1}\,\mathrm{id}_{\tilde{V}}\right) = \left(J + \sqrt{-1}\,\mathrm{id}_{\tilde{V}}\right)\tilde{V}$$

by W_1 and denote the $\mathbb{C}$-vector subspace

$$\operatorname{Ker}\left(\tilde{J} + \sqrt{-1}\,\mathrm{id}_{\tilde{V}}\right) = \left(J - \sqrt{-1}\,\mathrm{id}_{\tilde{V}}\right)\tilde{V}$$

by W_2 so that $\tilde{V} = W_1 \oplus W_2$ and $\overline{W_1} = W_2$. Let $\tilde{g}_{jk} : W_j \times W_k \to \mathbb{C}$ for $1 \leq j,k \leq 2$ be the restriction of $\tilde{g} : \tilde{V} \times \tilde{V} \to \mathbb{C}$. Let $h_1 : W_1 \times W_1 \to \mathbb{C}$ be defined by $h_1(u,v) = \tilde{g}_{12}(u,\bar{v})$ for $u,v \in W_1$. Let $h_2 : W_2 \times W_2 \to \mathbb{C}$ be defined by $h_1(u,v) = \tilde{g}_{21}(u,\bar{v})$ for $u,v \in W_2$.

Show that $g : V \times V \to \mathbb{R}$ satisfies $g(Jv,Jv) = g(v,v)$ for all $v \in V$ if and only if the following four conditions hold.

(i) $g_{11} : W_1 \times W_1 \to \mathbb{C}$ is the zero map.

(ii) $g_{22} : W_2 \times W_2 \to \mathbb{C}$ is the zero map.

(iii) $h_1 : W_1 \times W_1 \to \mathbb{C}$ defines an inner product of the $\mathbb{C}$-vector space W_1.

(iv) $h_2 : W_2 \times W_2 \to \mathbb{C}$ defines an inner product of the $\mathbb{C}$-vector space W_2.

Moreover, show that in such a case

$$g(w+\bar{w}, w+\bar{w}) = 2h_1(w,w) = 2h_2(\bar{w},\bar{w}) \quad \text{for } w \in W_1.$$

(*Hint:* consider the action of $\tilde{J}$ on W_j from the definition of W_j for $j = 1,2$ and express $g(u,v)$ as a linear combination of $g(w_k,w_k)$ for some suitable elements w_k of V with some universal constants as coefficients.)

Problem 5. (Minimal polynomials and direct sum decompositions) Let V be a vector space over a field $\mathbb{F}$ of positive finite dimension n. Let $T \in \operatorname{End}_{\mathbb{F}}(V)$ be non identically zero. For a subset A of V a polynomial $P(X) \in \mathbb{F}[X]$ is called an *annihilating polynomial* for T on A if $P(T)v = 0$ for every $v \in A$. When A consists of a single nonzero $v \in V$, we say that $P(X)$ is an annihilating polynomial for T at v if $P(T)v = 0$.

(a) Show that there is a nonzero annihilating polynomial for T on all of V. (*i.e.,* there exists a nonzero polynomial $P(X) \in \mathbb{F}[X]$ such that $P(T)V = 0$). (*Hint:* for some nonzero element v of V consider the infinite sequence $T^k v$ for $k \in \mathbb{N}$. Use the finite dimensionality of V and induction on the dimension of V and quotient vector spaces.)

(b) Let $\mathcal{P}$ be the set of all annihilating polynomials for T on A. Show that there is, uniquely up to multiplication by a nonzero element of $\mathbb{F}$, an element $Q(X)$ of $\mathcal{P}$ which divides every element of $\mathcal{P}$. We call $Q(X)$ a *minimal annihilating polynomial* for T on A.

(c) For a nonzero $v \in V$, let W be the smallest $\mathbb{F}$-vector subspace of V such that $v \in W$ and $TW \subset W$. Let $Q(X)$ be a minimal annihilating polynomial for T at v. Show that the degree of $Q(X)$ is no more than the dimension of W over $\mathbb{F}$. (*Hint:* apply the linear dependence lemma to $v, Tv, T^2v, T^3v \cdots$.)

(d) Show that the degree of a minimal annihilating polynomial for T on all of V is no more than n. (*Hint:* use Part (c) and quotient vector spaces.)

(e) Let $P(X) \in \mathbb{F}[X]$ be a minimal annihilating polynomial for T on all of V. Let

$$P(X) = (P_1(X))^{k_1} (P_2(X))^{k_2} \cdots (P_\ell(X))^{k_\ell}$$

be a factorization into products of irreducible polynomials

$$P_1(X), P_2(X), \cdots, P_\ell(X) \in \mathbb{F}[X]$$

with $k_j \in \mathbb{N}$ for $1 \leq j \leq \ell$. For $1 \leq j \leq \ell$ let W_j be the set of all $v \in V$ such that $(P_j(T))^{k_j} v = 0$. Show that

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_\ell$$

and each W_j is nonzero for $1 \leq j \leq \ell$ and, as a matter of fact, $(P_j(T))^k W_j \neq \{0\}$ for $0 \leq k < k_j$.

Problem 6. (Finite dimensional analogue of Hodge decomposition) Let U, V, and W be three $\mathbb{C}$-vector spaces of positive finite dimension with inner products $\langle \cdot, \cdot \rangle_U$, $\langle \cdot, \cdot \rangle_V$, and $\langle \cdot, \cdot \rangle_W$ respectively. Let $T \in \mathrm{Hom}_{\mathbb{C}}(U, V)$ and

$S \in \mathrm{Hom}_{\mathbb{C}}(V, W)$ such that $ST = 0$ as an element of $\mathrm{Hom}_{\mathbb{C}}(U, W)$. Consider the element $SS^* + T^*T$ of $\mathrm{End}_{\mathbb{C}}(V)$, where $S^* \in \mathrm{Hom}_{\mathbb{C}}(V, U)$ and $T^* \in \mathrm{Hom}_{\mathbb{C}}(W, V)$ are respectively the adjoints of S and T with respect to the inner products $\langle \cdot, \cdot \rangle_U$, $\langle \cdot, \cdot \rangle_V$, and $\langle \cdot, \cdot \rangle_W$. Let $H = \mathrm{Ker}\,(SS^* + T^*T)$ be the $\mathbb{C}$-vector subspace of V which is the null space of $SS^* + T^*T$.

(a) Show that the inclusion map $H \to V$ induces a well-defined $\mathbb{C}$-linear map from H to $\mathrm{Ker}\, T\,/\mathrm{Im}\, S$ which is an isomorphism.

(b) Show that $V = H \oplus \mathrm{Im}\, S \oplus \mathrm{Im}\, T^*$ and that the three $\mathbb{C}$-vector subspaces H, $\mathrm{Im}\, S$, and $\mathrm{Im}\, T^*$ of V are mutually orthogonal with respect the inner product $\langle \cdot, \cdot \rangle_V$ of V.