Math 55a, Assignment #9, November 14, 2003

Notations. $\mathbb{R}$ is the field of all real numbers. $\mathbb{C}$ is the field of all complex numbers. $\mathbb{N}$ denotes the set of all natural numbers (*i.e.,* all positive integers). For a field $\mathbb{F}$ and $\mathbb{F}$-vector spaces V and W, $\mathrm{Hom}_{\mathbb{F}}(V,W)$ denotes the set of all $\mathbb{F}$-linear maps from V to W and $\mathrm{End}_{\mathbb{F}}(V)$ denotes the set of all $\mathbb{F}$-linear maps from V to itself. The dual $\mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$ of V is denoted by V^*. The dual $(V^*)^*$ of the dual V^* of V is naturally identified with V. For $T \in \mathrm{Hom}_{\mathbb{F}}(V,W)$, $T^* \in \mathrm{Hom}_{\mathbb{F}}(W^*,V^*)$ denotes the adjoint of T. The set of all $\mathbb{F}$-bilinear $\mathbb{F}$-valued functions on $V^* \times W^*$ is denoted by $V \otimes W$.

For $k \in \mathbb{N}$ the set of all $\mathbb{F}$-multi-linear $\mathbb{F}$-valued functions on

$$\underbrace{V \times \cdots \times V}_{k \text{ copies}}$$

is denoted by

$$\underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}}$$

or $V^{\otimes k}$. The set of all skew-symmetric $\mathbb{F}$-multi-linear $\mathbb{F}$-valued functions on

$$\underbrace{V \times \cdots \times V}_{k \text{ copies}}$$

is denoted by

$$\underbrace{V \wedge \cdots \wedge V}_{k \text{ copies}}$$

or $\wedge^k V$. (A function is skew-symmetric if its sign is changed whenever any two of its arguments are switched.) The set $\wedge^k V$ is an $\mathbb{F}$-vector subspace of $V^{\otimes k}$.

Every $T \in \mathrm{End}_{\mathbb{F}}(V)$ induces element of $\mathrm{End}_{\mathbb{F}}(V^{\otimes k})$ which we denote by $T^{\otimes k}$. We denote by $\wedge^k T$ the element of $\mathrm{End}_{\mathbb{F}}(\wedge^k V)$ which is the restriction of $T^{\otimes k}$ to $\wedge^k V$.

For $v_1, v_2 \in V = \mathrm{Hom}_{\mathbb{F}}(V^*, \mathbb{F})$ the wedge product $v_1 \wedge v_2$ is the element of $\wedge^2 V$ which is defined by the skew-symmetric $\mathbb{F}$-bilinear $\mathbb{F}$-valued function

$$(u_1^*, u_2^*) \mapsto \frac{1}{2}\left(v_1(u_1^*)v_2(u_2^*) - v_2(u_1^*)v_1(u_2^*)\right)$$

on $V^* \times V^*$. Likewise, for $v_1, \cdots, v_k \in V = \mathrm{Hom}_{\mathbb{F}}(V^*, \mathbb{F})$ the wedge product $v_1 \wedge \cdots \wedge v_k$ is the element of $\wedge^k V$ defined by the skew-symmetric $\mathbb{F}$-multilinear $\mathbb{F}$-valued function on

$$\underbrace{V \times \cdots \times V}_{k \text{ copies}}$$

which is obtained by skew-symmetrizing

$$(u_1^*, \cdots, u_k^*) \mapsto v_1(u_1^*) \cdots v_k(u_k^*).$$

The determinant of a matrix A is denoted by $\det A$.

Problem 1. (Laplace expansion of determinant) Let V be a vector space over a field $\mathbb{F}$ of dimension n. Let $e_1, \cdots, e_n$ be an $\mathbb{F}$-basis of V. Let $1 \leq m < n$. Let $T \in \mathrm{End}_{\mathbb{F}}(V)$ whose matrix with respect to $e_1, \cdots, e_n$ is

$$C = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

For $j_1 < j_2 < \cdots < j_m$ and $j_{m+1} < j_{m+2} < \cdots < j_n$ with $\{j_1, j_2, \cdots, j_n\} = \{1, 2, \cdots, n\}$, let

$$A_{j_1, j_2, \cdots, j_m} = \begin{pmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_m} \\ a_{2j_1} & a_{2j_2} & \cdots & a_{2j_m} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{mj_1} & a_{mj_2} & \cdots & a_{mj_m} \end{pmatrix}$$

and

$$B_{j_1, j_2, \cdots, j_m} = \begin{pmatrix} a_{m+1\, j_{m+1}} & a_{m+2\, j_{m+2}} & \cdots & a_{m+1\, j_n} \\ a_{m+2\, j_{m+1}} & a_{m+2\, j_{m+2}} & \cdots & a_{m+2\, j_n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n\, j_{m+1}} & a_{n\, j_{m+2}} & \cdots & a_{n\, j_n} \end{pmatrix}.$$

Let $\operatorname{sign}(j_1, j_2, \cdots, j_n)$ be the sign of the permutation $(j_1, j_2, \cdots, j_n)$ of $(1, 2, \cdots, n)$. Show that

$$\det C = \sum_{j_1, \cdots, j_m} \operatorname{sign}(j_1, j_2, \cdots, j_n) \left(\det A_{j_1, j_2, \cdots, j_m}\right) \left(\det B_{j_1, j_2, \cdots, j_m}\right),$$

where the summation $\sum_{j_1, \cdots, j_m}$ is over all permutations $(j_1, j_2, \cdots, j_n)$ of $(1, 2, \cdots, n)$ with $j_1 < j_2 < \cdots < j_m$ and $j_{m+1} < j_{m+2} < \cdots < j_n$. (*Hint:* compare with the proof of the expansion of a determinant according to a row.)

Problem 2. Let V be an $\mathbb{F}$-vector space of finite dimension n. Let $e_1, \cdots, e_n$ be an $\mathbb{F}$-basis of V and $e_1^*, \cdots, e_n^* \in V^*$ be its dual basis. Let $T \in \operatorname{End}_{\mathbb{F}}(V)$ and $T^* \in \operatorname{End}_{\mathbb{F}}(V^*)$ be its adjoint.

(a) Prove that the determinant of the matrix of T with respect to $e_1, \cdots, e_n$ is equal to the determinant of the matrix of T^* with respect to $e_1^*, \cdots, e_n^*$.

(b) Fix an $\mathbb{F}$-isomorphism $\Phi : \wedge^n V \to \mathbb{F}$. Consider the pairing

$$\Xi : V \times \left(\wedge^{n-1} V\right) \to \mathbb{F}$$

defined by

$$(v_1,\, v_2 \wedge \cdots \wedge v_n) \mapsto \Phi\left(v_1 \wedge \cdots \wedge v_n\right).$$

Let $(a_{i,j})_{1 \le i,j \le n}$ be the matrix of T with respect to $e_1, \cdots, e_n$. For $1 \le i, j \le n$ let $A_{j,i}$ be the $(n-1) \times (n-1)$-determinant obtained by removing the i-th row and the j-th column of the matrix $(a_{i,j})_{1 \le i,j \le n}$. Let $(b_{j,i})_{1 \le j,i \le n}$ be the matrix whose (j,i)-th element is $(-1)^{i+j} A_{j,i}$. Show that the determinant of $(b_{j,i})_{1 \le j,i \le n}$ is equal to the determinant of $(a_{i,j})_{1 \le i,j \le n}$ raised to power $n-1$. (*Hint:* consider the matrix of $T^{\wedge(n-1)} \in \operatorname{End}_{\mathbb{F}}(\wedge^{n-1} V)$ with respect to the basis

$$e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_n \quad \text{for } 1 \le j \le n$$

of $\wedge^{n-1} V$; apply Part (a) to the pairing Ξ; and use the pairing Ξ to compare the basis

$$e_1 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_n \quad \text{for } 1 \le j \le n$$

of $\wedge^{n-1} V$ with the dual basis $e_1^*, \cdots, e_n^*$ of V).

Problem 3. Let $1 \leq m < n$. Let

$$C = \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

For $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ and $1 \leq k_1 < k_2 < \cdots < k_m \leq n$ let

$$A_{j_1, j_2, \cdots, j_m; k_1, k_2, \cdots, k_m} = \det \begin{pmatrix} a_{j_1,k_1} & a_{j_1,k_2} & \cdots & a_{j_1,k_m} \\ a_{j_2,k_1} & a_{j_2,k_2} & \cdots & a_{j_2,k_m} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{j_m,k_1} & a_{j_m,k_2} & \cdots & a_{j_m,k_m} \end{pmatrix}$$

Consider $(j_1, j_2, \cdots, j_m)$ with $1 \leq j_1 < j_2 < \cdots < j_m \leq n$ as a single index. There are $\binom{n}{m}$ such indices, where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

is the binomial coefficient. Let B be the determinant of order $\binom{n}{m}$ whose entry in the position

$$((j_1, j_2, \cdots, j_m), (k_1, j_2, \cdots, k_m))$$

is $A_{j_1, j_2, \cdots, j_m; k_1, j_2, \cdots, k_m}$. Express B in terms of C and n and m. (*Hint:* let $T \in \mathrm{End}_{\mathbb{F}}(\mathbb{F}^n)$ whose matrix is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

with respect to the natural basis $e_1, \cdots, e_n$ of $\mathbb{F}^n$. Compare the matrix of $T^{\wedge k} \in \mathrm{End}_{\mathbb{F}}\left(\wedge^k(\mathbb{F}^n)\right)$ with the matrix whose entry in the position

$$((j_1, j_2, \cdots, j_m), (k_1, j_2, \cdots, k_m))$$

is $A_{j_1,j_2,\cdots,j_m;k_1,j_2,\cdots,k_m}$ when the basis

$$e_{j_1}\wedge\cdots\wedge e_{j_m}\quad(\text{for } 1\leq j_1<\cdots<j_m\leq n)$$

of $\wedge^k(\mathbb{F}^n)$ is used. Consider upper triangular forms.)

Problem 4

(a) Verify that the following three Pauli's matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad\sigma_2=\begin{pmatrix}0&-\sqrt{-1}\\\sqrt{-1}&0\end{pmatrix},\quad\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

are all unitary (*i.e.* isometries in $\mathbb{C}^2$) and are square roots of the 2×2 identity matrix.

(b) Verify that each of the following four Eddington's matrices

$$\begin{pmatrix}\sqrt{-1}\,\sigma_1&0\\0&0\end{pmatrix},\quad\begin{pmatrix}\sqrt{-1}\,\sigma_3&0\\0&\sqrt{-1}\,\sigma_3\end{pmatrix},$$
$$\begin{pmatrix}0&-\sigma_2\\\sigma_2&0\end{pmatrix},\quad\begin{pmatrix}-\sqrt{-1}\,\sigma_2&0\\0&\sqrt{-1}\,\sigma_2\end{pmatrix}$$

is equal to the negative of its complex-conjugate transpose and is unitary and also is the square root of the negative of the 4×4 identity matrix.

Problem 5. Consider each of the n^2 entries of an $n\times n$ matrix $X=(x_{j,k})_{1\leq j,k\leq n}$ as a variable. Write the characteristic polynomial $\det(X-\lambda I)$ in the form $\sum_{\ell=0}^n f_\ell(X)\lambda^{n-\ell}$, where $f_\ell(X)$ means a function of the n^2 entries of X.

(a) Show that $f_k(AB)=f_k(BA)$ for any pair of $n\times n$ matrices A and B. (Note that this is the generalization of the statements for the trace and the determinant.) (*Hint:* continuity arguments reduce the general case to the case of invertible A and B.)

(b) Conversely, if $\phi(X)$ is a polynomial in the n^2 entries of X and has the property that $\phi(AB)=\phi(BA)$ for any pair of $n\times n$ matrices A and B, then show that there exists a polynomial $P(Y_1,\cdots,Y_n)$ of n variables such that $\phi(X)=P\left(f_1(X),\cdots,f_n(X)\right)$ for any $n\times n$ matrix X. (*Hint:* continuity arguments reduce the general case to the case of X having distinct eigenvalues; diagonalize X; use the fact that a polynomial symmetric in its variables is a polynomial of elementary symmetric functions.)

Problem 6. Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ k & 0 & 2 & 0 \\ 0 & k & 0 & 2 \\ 0 & 0 & k & 0 \end{pmatrix},$$

where k is a constant.

(a) Find a value of k such that the matrix A is diagonalizable.

(b) Find a value of k such that the matrix A is not diagonalizable.