Math 55a, Quiz #1 Solutions, October 1, 2003

Notations. $\mathbb{N}$ = the set of all natural numbers (*i.e.,* all positive integers).
$\mathbb{Q}$ = the set of all rational numbers (*i.e.,* all quotients of integers with nonzero denominators).
$\mathbb{C}$ = the set of all complex numbers.

Problem 1. Let X be a metric space and $a \in X$. Let E be the subset of X defined as follows. A point $x \in X$ belongs to E if and only if there exists a *connected* subset F of X (which may depend on x) such that both x and a belong to F. Show that E is connected.

Solution of Problem 1. Suppose the contrary. Then E is the union of two disjoint nonempty open subsets A and B of E. We can assume without loss of generality that $a \in A$ (by renaming A and B if necessary). Since B is nonempty, there exists $b \in B$. By definition of E it follows from $b \in B \subset E$ that there exists a connected subset C of X such that both a and b belong to C. Every point c in C belongs to E, because both a and c belong to the connected subset C of X. The two sets $A \cap C$ and $B \cap C$ (which contain respectively a and b) are both nonempty disjoint open subsets of C. Moreover, the union of $A \cap C$ and $B \cap C$ is C, because $C \subset E = A \cup B$. This contradicts the fact that C is connected.

Problem 2. Let X be a nonempty compact metric space with metric $d(\cdot,\cdot)$. Let $0 < c < 1$ and let $T : X \to X$ be a map such that $d(T(x), T(y)) \leq c\, d(x,y)$ for $x, y \in X$. Show that there exists some $x_0 \in X$ such that $Tx_0 = x_0$.

Solution of Problem 2. Since X is nonempty, there exists $y_0 \in X$. If $Ty_0 = y_0$, we can take $x_0 = y_0$. We now assume $Ty_0 \neq y_0$. Let $a = d(y_0, Ty_0)$. Then $a > 0$. Let Y be the set $\left\{T^k y_0\right\}_{k\in\mathbb{N}}$. Choose $x_0 \in X$ in the following way. If Y is finite, we choose x_0 which is equal to y_k for infinite number of distinct $k \in \mathbb{N}$. If Y is infinite, by the compactness of X every infinite set has a limit point and we choose x_0 to be a limit point of Y. Then for every $\varepsilon > 0$ and every $N \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with $k \geq N$ such that $d\left(x_0, T^k y_0\right) < \varepsilon$. We claim that $Tx_0 = x_0$. Assume the contrary. Let $b = d(x_0, Tx_0)$. Then $b > 0$. Choose $\varepsilon > 0$ such that $\varepsilon < \frac{b}{2(1+c)}$. Choose $N \in \mathbb{N}$ such that $c^N a < \frac{b}{2}$. There exists $k \in \mathbb{N}$ with $k \geq N$ such that $d\left(x_0, T^k y_0\right) < \varepsilon$. We can write

$$
\begin{aligned}
b = d\left(x_0, T x_0\right) &\leq d\left(x_0, T^k y_0\right) + d\left(T^k y_0, T^{k+1} y_0\right) + d\left(T^{k+1} y_0, T x_0\right) \\
&\leq \varepsilon + c^k a + c\varepsilon \leq \varepsilon\left(1 + c\right) + c^N a < \frac{b}{2} + \frac{b}{2} = b,
\end{aligned}
$$

which is a contradiction.

Problem 3. Let $X \subset \mathbb{C}^{\mathbb{N}}$ be the set of maps $f : \mathbb{N} \to \mathbb{C}$ such that

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{n} |f(k)|$$

exists as an element of $\mathbb{R}$. Define the metric

$$d\left(f, g\right) = \sup_{n \in \mathbb{N}} \sum_{k=1}^{n} |f(k) - g(k)|$$

on X so that X becomes a metric space.

(a) Let E be the set of all $f \in X$ with $d(f, 0) \leq 1$, where $0 \in X$ means the element whose value at every $n \in \mathbb{N}$ is zero. In other words, E consists of all $f \in X$ with

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{n} |f(k)| \leq 1.$$

Show that E is not compact.

(b) Let Y be the subset of X defined as follows. An element $f : \mathbb{N} \to \mathbb{C}$ of X belongs to Y if and only if there exists some $n \in \mathbb{N}$ (which may depend on f) such that $f(k) = 0$ for $k \geq n$ and both the real part and the imaginary part of $f(\ell)$ belongs to $\mathbb{Q}$ for $\ell < n$. Show that Y is a countable dense subset of X.

Solution of Part (a) of Problem 3. For $n \in \mathbb{N}$ let g_n be the element of X such that $g_n(k) = 0$ for $k \neq n$ and $g_n(n) = 1$ when g_n is considered as a map from $\mathbb{N}$ to X. Suppose E is compact. Then the infinite set $\left\{g_n\right\}_{n \in \mathbb{N}}$ has a limit point h in E (Theorem 2.37 on Page 38 of Rudin's book). The neighborhood

$$\left\{f \in E \;\middle|\; d\left(f, h\right) < 1\right\}$$

of h must contain infinitely many points of $\{g_n\}_{n\in\mathbb{N}}$ (Theorem 2.20 on Page 32 of Rudin's book), in particular, at least two distinct points of $\{g_n\}_{n\in\mathbb{N}}$. There exist $k \neq \ell$ in $\mathbb{N}$ such that $d(g_k, h) < 1$ and $d(g_\ell, h) < 1$. Since $k \neq \ell$, it follows that

$$d(g_k, g_\ell) = |g_k(k)| + |g_\ell(\ell)| = 2,$$

which contradicts

$$d(g_k, g_\ell) \leq d(g_k, h) + d(h, g_\ell) < 1 + 1 = 2.$$

Solution of Part (b) of Problem 3. For $n \in \mathbb{N}$ let Y_n be the set of all $f \in Y$ such that $f(k) = 0$ for $k \geq n$. For fixed n the set Y_n is a subset of the product of $n-1$ copies of $\mathbb{Q}$ and is therefore countable (use of Theorem 2.12 on Page 29 of Rudin's book $n-2$ times). Since Y is the union of Y_n for $n \in \mathbb{N}$, it follows that Y is also countable (Theorem 2.12 on Page 29 of Rudin's book). Take an arbitrary open neighborhood

$$N_h(\varepsilon) := \left\{ f \in X \;\middle|\; d(f, h) < \varepsilon \right\}$$

in X, where $h \in X$ and $\varepsilon > 0$. Let

$$A = \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} |h(k)|.$$

To show that Y is dense in X it suffices to verify that the intersection of Y and the neighborhood $N_h(\varepsilon)$ is nonempty. By the definition of supremum there exists some $N \in \mathbb{N}$ such that

$$\sum_{k=1}^{N} |h(k)| \geq A - \frac{\varepsilon}{2}.$$

Then for any $n \geq N+1$,

$$\sum_{k=N+1}^{n} |h(k)| = \sum_{k=1}^{n} |h(k)| - \sum_{k=1}^{N} |h(k)| \leq A - \left(A - \frac{\varepsilon}{2}\right) = \frac{\varepsilon}{2}.$$

For every $\ell \in \mathbb{N}$ with $\ell \leq N$ we choose $c_\ell \in \mathbb{C}$ whose real part and imaginary part are both in $\mathbb{Q}$ and

$$|h(\ell) - c_\ell| < \frac{\varepsilon}{2N}.$$

Define an element $g : \mathbb{N} \to \mathbb{C}$ of Y by setting $g(k) = 0$ for $k > N$ and $g(\ell) = c_\ell$ for $\ell \leq N$. It follows from

$$\sum_{k=1}^{n} |g(k) - h(k)| \leq \sum_{k=1}^{N} |g(k) - h(k)| + \sum_{\ell=N+1}^{n} |h(\ell)| < N \frac{\varepsilon}{2N} + \frac{\varepsilon}{2} = \varepsilon$$

for any $n \in \mathbb{N}$ that $d(g,h) < \varepsilon$ and g belongs to the neighborhood $N_h(\varepsilon)$. Here we use the notational convention that $\sum_{\ell=N+1}^{n} |h(\ell)|$ means 0 when $n \leq N$.