**Take-Home Final Examination of Math 55a**
**(January 17 to January 23, 2004)**

N.B. For problems which are similar to those on the homework assignments, complete self-contained solutions are required and homework problems cannot be quoted simply as known facts in the solutions.

*Notations.* $\mathbb{N}$ = all positive integers.
$\mathbb{Z}$ = all integers.
$\mathbb{R}$ = all real numbers.
$\mathbb{C}$ = all complex integers.
$\mathbb{F}$ means either $\mathbb{C}$ or $\mathbb{R}$.

*Problem 1.* The five axioms of Peano are the following.

(1) The set $\mathbb{N}$ of all natural numbers contains an element 1.

(2) There is an *immediate successor* $x' \in \mathbb{N}$ defined for every element $x \in \mathbb{N}$.

(3) 1 is not an immediate successor of any element of $\mathbb{N}$.

(4) Two distinct elements of $\mathbb{N}$ have distinct immediate successors.

(5) If a subset $E$ of $\mathbb{N}$ contains 1 and contains the immediate successor of every one of its elements, then $E$ must be all of $\mathbb{N}$.

Addition in $\mathbb{N}$ is defined by $x+1 = x'$ and $x+y' = (x+y)'$. Multiplication in $\mathbb{N}$ is defined by $x \cdot 1 = x$ and $x \cdot y' = (x \cdot y) + x$. From the five Peano's axioms and the definitions of addition and multiplication prove that $x \cdot (y+z) = (x \cdot y) + (x \cdot z)$ for $x, y, z \in \mathbb{N}$.

*Problem 2.* For every natural number $\nu \in \mathbb{N}$ let $X_\nu$ be a (nonempty) metric space with metric $d_{X_\nu}(\cdot,\cdot)$. Let $\mathcal{X}$ be the product space $\prod_{\nu \in \mathbb{N}} X_\nu$. We denote the components of an element $\mathbf{x} \in \mathcal{X}$ by $x_\nu$ so that we write $\mathbf{x} = \{x_\nu\}_{\nu=1}^\infty$ with $x_\nu \in X_\nu$. Let $\rho > 1$. Define the metric $d_{\mathcal{X}}(\cdot,\cdot)$ on $\mathcal{X}$ by

$$d_{\mathcal{X}}\left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right) = \sum_{\nu=1}^\infty \frac{1}{\rho^\nu} \frac{d_{X_\nu}\left(x_\nu^{(1)}, x_\nu^{(2)}\right)}{1 + d_{X_\nu}\left(x_\nu^{(1)}, x_\nu^{(2)}\right)}.$$

for $\mathbf{x}^{(1)} = \left\{x_\nu^{(1)}\right\}_{\nu=1}^\infty$ and $\mathbf{x}^{(2)} = \left\{x_\nu^{(2)}\right\}_{\nu=1}^\infty$.

(a) Verify that $d_{\mathcal{X}}$ is indeed a metric on $\mathcal{X}$.

(b) Verify that a subset $G$ of $\mathcal{X}$ is open in $\mathcal{X}$ if and only if for every point $\mathbf{x}^{(0)} = \left\{x_\nu^{(0)}\right\}_{\nu=1}^{\infty}$ of $G$ there exist some $N \in \mathbb{N}$ and some positive numbers $r_1, r_2, \cdots, r_N$ such that every point $\mathbf{x} = \{x_\nu\}_{\nu=1}^{\infty}$ of $\mathcal{X}$ with $d_{X_\nu}\left(x_\nu, x_\nu^{(0)}\right) < r_\nu$ for $1 \leq \nu \leq N$ belongs to $G$.

(c) Show that $\mathcal{X}$ is compact if and only if each $X$ is compact.

*Problem 3.* Let $X$ and $Y$ be metric spaces and $f : X \to Y$ be a *surjective* continuous map. Assume the following three conditions.

(a) $X$ is compact.

(b) $f^{-1}(y)$ is connected for every $y \in Y$.

(c) $Y$ is connected.

Prove that $X$ is connected.

*Problem 4.* Let $c_n$ for $n \in \mathbb{N}$ be a *non-increasing* sequence of positive numbers. Prove that the following two statements are equivalent.

(a) For any $-\infty < a < b < \infty$ the sequence

$$\sum_{n=1}^{\infty} c_n \sin nx$$

converges uniformly on $[a, b]$ (in the sense that given any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$\left|\sum_{n=p}^{q} c_n \sin nx\right| < \varepsilon$$

for $a \leq x \leq b$ and $p,\, q \geq N$).

(b)

$$\lim_{n\to\infty} nc_n = 0.$$

(*Hint:* For $(a) \Rightarrow (b)$, for any $n$ sufficiently large choose $p$ roughly of the order $\frac{n}{2}$ and choose $x$ positive sufficiently close to zero, roughly of the order $\frac{\pi}{n}$, such that $\sum_{k=p}^{n} c_k \sin kx$ dominates a fixed positive number times $nc_n$. For $(b) \Rightarrow (a)$, argue as follows. For $x \geq \frac{\pi}{p}$, use summation by parts and bound $c_p \sum_{n=p}^{q} \sin nx$ by using the summation formula for $\sum_{n=p}^{q} \sin nx$. For $x \leq \frac{\pi}{q}$, use $\sin\theta < \theta$ for $\theta > 0$ to bound $\sum_{n=p}^{q} c_n \sin nx$. For $\frac{\pi}{q} < x < \frac{\pi}{p}$, bound $\sum_{n=p}^{q} c_n \sin nx$ by breaking it up suitably into two summands and use separately the preceding two bounding arguments for the two summands.)

*Problem 5.* Let $V$ be a vector space over $\mathbb{F}$ of finite dimension $n$ which is endowed with an inner product $\langle \cdot, \cdot \rangle$. Let $T : V \to V$ be an $\mathbb{F}$-linear map which is self-adjoint with respect to $\langle \cdot, \cdot \rangle$ (that is, $\langle Tv, w \rangle = \langle v, Tw \rangle$ for all $v, w \in V$). Consider the following procedure. Choose a vector $v_1$ of unit length in $V$ such that $\langle Tv, v \rangle$ achieves its minimum at $v = v_1$ among all $v \in V$ of unit length. Inductively suppose $v_1, \cdots, v_k$ have been chosen and the set $v_1, \cdots, v_k$ does not span $V$ over $\mathbb{F}$. Let $V_k$ be the orthogonal complement of the $\mathbb{F}$-vector subspace of $V$ spanned by $v_1, \cdots, v_k$. Choose a vector $v_{k+1}$ of unit length in $V_k$ such that $\langle Tv, v \rangle$ achieves its minimum at $v = v_{k+1}$ among all $v \in V_k$ of unit length. Show that this procedure produces an orthonormal basis $v_1, \cdots, v_n$ with respect to which $T$ is represented by a diagonal matrix with real eigenvalues. Justify carefully each step and explain why each $v_k$ exits.

*Problem 6.* Let $V$ be a vector space over $\mathbb{F}$ of finite dimension $n$. Denote by $V^*$ the dual vector space of $V$ and regard $V$ as the set of all $\mathbb{F}$-valued $\mathbb{F}$-linear functions on $V^*$. Let $1 \leq k \leq n$. Define the *exterior product* $\wedge^k V$ of $k$ copies of $V$ as the set of all $\mathbb{F}$-valued $\mathbb{F}$-multilinear functions on

$$\underbrace{V^* \times V^* \times \cdots \times V^*}_{k \text{ copies}}$$

which are skew-symmetric in its $k$ variables (that is, the value of the function changes sign when any two of the $k$ variables are interchanged). For $v_1, \cdots, v_k \in V$ define the *wedge product* $v_1 \wedge \cdots \wedge v_k$ as the $\mathbb{F}$-valued $\mathbb{F}$-multilinear function on

$$\underbrace{V^* \times V^* \times \cdots \times V^*}_{k \text{ copies}}$$

which is the skew-symmetrization of the function

$$(x_1, x_2, \cdots, x_k) \mapsto v_1(x_1)\, v_2(x_2) \cdots v_k(x_k)$$

for $x_1, x_2, \cdots, x_k \in V^*$. In other words,

$$\begin{aligned}&\left(v_1 \wedge v_2 \wedge \cdots \wedge v_k\right)\left(x_1, x_2, \cdots, x_k\right)\\ &= \frac{1}{k!} \sum_{\sigma} \operatorname{sign}(\sigma)\, v_1\left(x_{\sigma(1)}\right) v_2\left(x_{\sigma(2)}\right) \cdots v_k\left(x_{\sigma(k)}\right),\end{aligned}$$

where the summation is over all the $k!$ permutations $\sigma$ of the $k$ letters $\{1, 2, \cdots, k\}$ and $\operatorname{sign}(\sigma)$ is the signature of the permutation $\sigma$. Let $\langle\cdot,\cdot\rangle_V$ be an inner product of $V$. Let $e_1, \cdots, e_n$ be an orthonormal basis of $V$ over $\mathbb{F}$. Let $\langle\cdot,\cdot\rangle_{\wedge^k V}$ be the inner product on $\wedge^k V$ which is defined by the condition that the following collection of $\binom{n}{k}$ elements of $\wedge^k V$

$$e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k} \quad (1 \leq j_1 < j_2 < \cdots < j_k \leq n)$$

form an *orthonormal* basis of $\wedge^k V$ over $\mathbb{F}$. Show that for $u_1, \cdots, u_k \in V$ and $v_1, \cdots, v_k \in V$ the inner product

$$\left\langle u_1 \wedge \cdots \wedge u_k,\, v_1 \wedge \cdots \wedge v_k \right\rangle_{\wedge^k V}$$

of the two elements $u_1 \wedge \cdots \wedge u_k$ and $v_1 \wedge \cdots \wedge v_k$ of $\wedge^k V$ is equal to the determinant of the $k \times k$ matrix whose element on the $j$-th row and in the $\ell$-th column is $\langle u_j,\, v_\ell\rangle_V$.

*Hint:* Let $A$ be a $k \times n$ matrix and $B$ be an $n \times k$ matrix. For any $1 \leq j_1 < \cdots < j_k \leq n$ let $A_{j_1,\cdots,j_k}$ be the $k \times k$ matrix obtained from $A$ by taking only its $j$-th columns for $j = j_1, \cdots, j_k$. Let $B_{j_1,\cdots,j_k}$ be the $k \times k$ matrix obtained from $B$ by taking only its $j$-th rows for $j = j_1, \cdots, j_k$. Express the $k \times k$ determinant of $AB$ in terms of the collection of the determinants of $A_{j_1,\cdots,j_k} B_{j_1,\cdots,j_k}$ for all $1 \leq j_1 < \cdots < j_k \leq n$. The special case $k = 2$ of the problem for $u_1 = v_1 = \sum_{j=1}^n a_j e_j$ and $u_2 = v_2 = \sum_{j=1}^n b_j e_j$ is equivalent to the identity

$$\left(\sum_{j=1}^n |a_j|^2\right)\left(\sum_{j=1}^n |b_j|^2\right) - \left|\sum_{j=1}^n a_j \overline{b_j}\right|^2 = \sum_{1 \leq j < \ell \leq n} \left|a_k b_\ell - a_\ell b_k\right|^2,$$

which is used in the proof of the Cauchy-Schwarz inequality.

*Problem 7.* Let $-\infty < a < b < \infty$. For $n \in \mathbb{N}$ let $f_n(x)$ be a $\mathbb{C}$-valued continuous function on $[a, b]$ whose first-order derivative $f_n'(x)$ is also continuous on $[a, b]$. Assume that $|f_n(a)| \leq 1$ and

$$\int_a^b \left|f_n'(x)\right|^2 dx \leq 1$$

for $n \in \mathbb{N}$. Show that there is a subsequence $f_{n_j}$ ($j \in \mathbb{N}$) such that

$$\sup_{a \le x \le b} \left| f_{n_j}(x) - f_{n_k}(x) \right|$$

approach 0 as $j$, $k \to \infty$.

*Hint:* Use

$$\left| \int_x^y g(t)h(t)dt \right|^2 dt \le \left( \int_x^y |g(t)|^2 \, dt \right) \left( \int_x^y |h(t)|^2 \, dt \right)$$

for $x < y$ and use

$$f(x) - f(y) = \int_x^y f'(t)dt$$

to show that for $\varepsilon > 0$ the number $\delta > 0$ chosen in the definition for uniform continuity of $f_n(x)$ on $[a, b]$ can be chosen to be independent of $n$.)

*Problem 8.* Let $-\infty < a < b < \infty$. Let $X$ be the set of all $\mathbb{C}$-valued functions $f$ on $[a, b]$ which is continuous on $[a, b]$. Define the norm

$$\|f\|_X = \sup_{a \le x \le b} |f(x)|$$

for $f \in X$. Let $Y$ be the set of all $\mathbb{C}$-valued functions $g$ on $[a, b]$ which is continuous on $[a, b]$ and whose first-order derivative $g'$ is also continuous on $[a, b]$. Define the norm

$$\|g\|_Y = \sup_{a \le x \le b} (|g(x)| + |g'(x)|)$$

for $g \in Y$.

(a) Verify that $X$ with the norm $\|\cdot\|_X$ is a Banach space.

(b) Verify that $Y$ with the norm $\|\cdot\|_Y$ is a Banach space.

(c) Show that for every sequence $g_\nu$ in $Y$ ($\nu \in \mathbb{N}$) with $\|g_\nu\|_Y \le 1$ there is a subsequence $g_{\nu_j}$ ($j \in \mathbb{N}$) such that as a sequence in $X$ the subsequence $g_{\nu_j}$ converges in $X$ to some element of $X$ as $j \to \infty$.

(*Hint:* for the proof of (c) compare with Problem 7.)

*Problem 9.* For $0 < x < 1$ and $n \in \mathbb{N}$ let $f_n(x)$ be the distance between $x$ and the nearest number of the form $\frac{m}{10^n}$, where $m \in \mathbb{Z}$. Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Prove the following two statements.

(a) The function $f(x)$ is continuous at every point of $(0, 1)$.

(b) The function $f(x)$ is *not* differentiable at *any* point of $(0, 1)$.

(*Hint:* For the proof of (b), for a fixed $x \in (0, 1)$ let

$$x = \sum_{q=1}^{\infty} \frac{a_q}{10^q},$$

where $a_q \in \mathbb{Z}$ with $0 \le a_q \le 9$. Define $x_q = x - \frac{1}{10^q}$ if $a_q = 4$ or $9$, otherwise define $x_q = x + \frac{1}{10^q}$. Then

$$\frac{f(x_q) - f(x)}{x_q - x} = q',$$

where $q'$ is an integer which is congruent to $q - 1$ modulo 2.)

*Problem 10.* Suppose $-\infty < a < b < \infty$. Let $f(x)$ be a bounded real-valued function on $[a, b]$ and $\alpha(x)$ be a real-valued non-decreasing function $[a, b]$. Let $E$ be a subset of $[a, b]$. Assume the following two conditions.

(a) $f$ is continuous at every point of $[a, b]$ which is not in $E$.

(b) For ever $\varepsilon > 0$ there exist a finite number of disjoint open intervals $(c_1, d_1), \cdots, (c_N, d_N)$ inside $[a, b]$ such that $\sum_{j=1}^{N} (\alpha(d_j) - \alpha(c_j)) < \varepsilon$ and their union $\cup_{j=1}^{N} (c_j, d_j)$ contains $E$. (Note that in this condition the number $N$, as well as the intervals $(c_1, d_1), \cdots, (c_N, d_N)$, may depend on $\varepsilon$.)

Prove that $f$ is Riemann-Stieltjes integrable with respect to $\alpha$ on $[a, b]$ (that is, in the notation of the book of Rudin, $f \in \mathcal{R}(\alpha)$ on $[a, b]$).

*Problem 11.* For $1 < s < \infty$, define the *Riemann zeta function* by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Let $[x]$ denote the greatest integer $\le x$. Prove the following three statements.

(a)

$$\zeta(s) = s \lim_{b\to\infty} \int_{x=1}^{b} \frac{[x]}{x^{s+1}}\, dx\,.$$

(b)

$$\zeta(s) = \frac{s}{s-1} - s \lim_{b\to\infty} \int_{x=1}^{b} \frac{x-[x]}{x^{s+1}}\, dx\,.$$

(c) The limit

$$\lim_{b\to\infty} \int_{x=1}^{b} \frac{x-[x]}{x^{s+1}}\, dx$$

exists for all $s > 0$.

(*Hint:* To prove (a), compute the difference between the integral over $[1, N]$ and the $N$-th partial sum of the series that defines $\zeta(s)$.)

*Problem 12.* Let $-\infty < a < b < \infty$. Let $f(x)$ be a real-valued continuous function on $[a, b]$ and $\phi(x)$ be a non-increasing function on $[a, b]$ whose first-order derivative $\phi'(x)$ is continuous on $[a, b]$. Show that there exists $\xi \in [a, b]$ such that

$$\int_{x=a}^{b} f(x)\phi(x)dx = \phi(a) \int_{x=a}^{\xi} f(x)dx + \phi(b) \int_{x=\xi}^{b} f(x)dx.$$

(*Hint:* First reduce to the special case where $\phi(b) = 0$. Let $F'(x) = f(x)$ with $F(a) = 0$. Use $F(x)$ to apply integration by parts to $f(x)\phi(x)$ and estimate $F(x)$ by its supremum and infimum on $[a, b]$ and use the Intermediate Value Theorem.)