Math 55a, Fall 2004

---

7 th Assignment, due November 11

---

1. This problem explores the notions of topological group and continuous group action. By definition, a topological group is a pair $(G, \mathcal{T}_G)$ consisting of a group G and a topology $\mathcal{T}_G$ which makes multiplication (as map from $G \times G$ to G) and inversion (as map from G to G) continuous. A continuous action of a topological group G on a topological space X is an action of G on X such that the action map $a : G \times X \to X$ is a continuous map.
a) Show that every T_0 topological group has the T_2 property (Hint: you'll need to use the fact that multiplication and inversion are continuous at $(e, e) \in G \times G$ and at e, respectively).
b) For this part of the problem, Y is a topological space, Z a set, and $F : Y \to Z$ a surjective map. Define a family of subsets $\mathcal{T}_Z$ of Z as follows: $U \subset Z$ belongs to $\mathcal{T}_Z$ precisely when $F^{-1}(U)$ is open in Y. Show that $\mathcal{T}_Z$ is a topology for Z, that $F : Y \to Z$ is continuous, and that $(Z, \mathcal{T}_Z)$ has the T_1 property if and only if $F^{-1}\{z\}$ is closed for every $z \in Z$. The topology $\mathcal{T}_Z$ is called the *quotient toplogy* induced by the topology of Y.
c) Show: an open subgroup of a topological group is necessarily closed.
d) Let G be a topological group and $H \subset G$ a closed subgroup. Equip G/H with the quotient topology. Show that G/H is a T_1 space, and that the natural action of G on G/H is continuous.
e) With G and H as in c), show that the quotient topology on G/H is Hausdorff provided G is a Hausdorff group (Hint: given $g \in G$ and an open neighborhood U of the identity, there exists another open neighborhood V of e such that $gVg^{-1} \subset U$).
f) Now suppose $H \subset G$ is a closed normal subgroup. Show that G/H, equipped with the quotient topology, is a topological group.
g) Verify: the closure of $\{e\}$ in any topological group is a normal subgroup, and the quotient of G by this subgroup is a Hausdorff group.
h) Let $GL(2, \mathbb{R})$ denote the set of invertible 2×2 matrices with real entries, and $SL(2, \mathbb{R})$ the set of 2×2 matrices with real entries and determinant 1. View both as subsets of $\mathbb{R}^4 \cong$ set of all real 2×2 matrices, and equip them with the restricted topology. Show that both are topological groups, with matrix multiplication as the law of composition.

2. Exhibit an isomorphism between $SL(2\,, \mathbb{Z}/2\,\mathbb{Z}))$ – i.e., the group of 2×2 matrices with entries in the field $\mathbb{Z}/2\,\mathbb{Z}$ and determinant 1 – and S_3, the permutation group on 3 letters.