Math 55a, Fall 2004

11 th Assignment, due December 7

1. Let K be a field. One defines the formal derivative p' of a polynomial $p \in K[X]$ by requiring that $p \mapsto p'$ is a linear map from $K[X]$ to itself which sends X^n, for $n \in \mathbb{Z}_{\geq 0}$, to nX^{n-1}. One says that "p has a root of order exactly n" at $a \in K$ (or "p vanishes to order exactly n at a") if $(X-a)^n$ divides p, but $(X-a)^{n+1}$ does not.

a) Prove: if K has characteristic zero, a polynomial $p \in K[X]$ vanishes to order exactly n at $a \in K$ if and only if $p(a) = p'(a) = \cdots = p^{(n-1)}(a) = 0$, $p^{(n)}(a) \neq 0$.

b) Let K be a field of characteristic zero, $p \in K[X]$ an irreducible polynomial. Show that p has only simple zeroes (i.e., zeroes of order one) in any extension field of K (hint: p, p' must be relatively prime).

2. Let K be a field of characteristic zero, and L a finite extension of K – i.e., an extension field, of finite degree over K. Following the steps outlined below, prove the "Theorem of the Primitive Element": there exists an element $\zeta \in L$ such that $L = K[\zeta]$. You will find certain problems on earlier assignments relevant.

a) It is enough to prove the theorem when $L = K[\alpha, \beta]$ ($=_{\text{def}}$ smallest subring of L containing α, β), for some α, $\beta \in L$. From now on, assume that $L = K[\alpha, \beta]$.

b) Let p, $q \in K[X]$ be the monic irreducible polynomials vanishing, respectively, at α and β (recall problem #1c of the 10th assignment). Show: there exists an extension field E of $L = K[\alpha, \beta]$ such that both p and q split into products of linear factors in $E[X]$. Enumerate the roots of p as $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_r$, and those of q as $\beta = \beta_1, \beta_2, \ldots, \beta_s$.

c) There exists $c \in K$ such that $\alpha_i + c\beta_j \neq \alpha + c\beta$, for $1 \leq i \leq r$ and $2 \leq j \leq s$. Fix $c \in K$ with this property, and set $\zeta = \alpha + c\beta$.

d) Define $\tilde{p}$ by the identity $\tilde{p}(X) = p(\zeta - cX)$. Then $\tilde{p} \in K[\zeta]$, and $X - \beta$ is the greatest common divisor of $\tilde{p}$ and q in $K[\zeta]$ (hint: can $\tilde{p}$ and q be relatively prime? what are the roots of $\tilde{p}$ and q in E?).

e) Deduce that $\beta \in K[\zeta]$, and hence $K[\zeta] = K[\alpha, \beta]$.

f) Precisely where and how did you use the hypothesis that $\operatorname{char}(K) = 0$?

3. In this problem, V and W denote vector spaces over a field K, and V^* denotes the dual space of V. Show:

a) There exists a canonical (non-zero!) linear map $V^* \otimes W \to \operatorname{Hom}(V, W)$ (Hint: the map $V^* \times W \times V \to W$, $(\phi, w, v) \mapsto \langle \phi, v \rangle w$ is linear in each of the three arguments).

b) If V and W are finite dimensional, the linear map constructed in a) establishes a canonical isomorphism $V^* \otimes W \simeq \operatorname{Hom}(V, W)$.

c) What can you say about the canonical linear map $V^* \otimes W \to \operatorname{Hom}(V, W)$ when V and/or W are infinite dimensional?