Math 55a, Fall 2004

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12 th Assignment, due December 14

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1. Let K be a field of characteristic zero, $L \supset K$ a finite extension, of degree n over K, and $E \supset K$ another extension (which may or may not contain L). An "embedding of L into E over K" is a homorphism of fields $\phi : L \to E$ which is the identity on K. An "automorphism of L over K" is a isomorphism of fields $\sigma : L \to L$ which is the identity on K. Show:

a) Every homomorphism of fields $\sigma : L \to L$ which is the identity on K is necessarily invertible (i.e., is an automorphism).

b) The number of distinct embeddings of L into E over K is at most n.

c) There exists a finite extension E of K such that the number of distinct embeddings of L into E over K is exactly equal to n.

d) The following conditions on the extension L are equivalent:
i) For every extension E of L, and for every embedding σ of L into E over K, $\sigma(L) = L$;
ii) L is the splitting field of an irreducible polynomial in $K[X]$;
iii) Every irreducible polynomial in $K[X]$ which has a root in L splits into a product of linear factors in $L[X]$;
iv) The number of distinct automorphisms of L over K is equal to n.

If L satisfies these equivalent conditions, it is said to be a "normal extension" or "Galois extension" of K (for fields of finite characteristic, which we are not considering here, the notion of Galois extension is more restrictive than normality). By definition, the Galois group of a Galois extension $L \supset K$ is the group of all automorphism of L over K; notation: $\mathrm{Gal}(L/K)$. For the next part of this problem, suppose that L is a Galois extension of K, of degree n. If $H \subset \mathrm{Gal}(L/K)$ is a subgroup, let L^H denote the "fixed field" of H, i.e., $L^H = \{\, x \in L \mid \sigma x = x \text{ for all } \sigma \in H \,\}$; note that L^H is indeed a field, and that it contains the ground field K.

e) The association $H \mapsto L^H$ establishes a bijection between the subgroups of $\mathrm{Gal}(L/K)$ and the subfields of L which contain K. The field L is a Galois extension of L^H, and $\mathrm{Gal}(L/L^H) = H$. The group H is normal in $\mathrm{Gal}(L/K)$ if and only if L^H is normal over K; if so, $\mathrm{Gal}(L^H/K) \simeq \mathrm{Gal}(L/K)\,/\,H$.

f) Determine $\mathrm{Gal}(L/\,\mathbb{Q}\,)$ for L = splitting field, over $\mathbb{Q}$, of the polynomial $X^4 + X^3 + X^2 + X + 1$ (hint: multiply the polynomial by $X - 1$).

g) Let ζ denote the positive cube root of 2. Is $\mathbb{Q}\,[\zeta]$ normal over $\mathbb{Q}$? If so, determine the Galois group of this extension.

2. A *representation* of a group G on a vector space V is a group homomorphism

$$\pi :\; G \;\longrightarrow\; \mathrm{Aut}(V) \;=_{\mathrm{def}}\; \{\, T \in \mathrm{End}(V) \;\mid\; T \text{ is invertible} \,\}\,.$$

Equivalently, it is the datum of a linear transformation $\pi(g) : V \to V$ for every $g \in G$, such that $\pi(e) = 1_V$ and $\pi(gh) = \pi(g) \circ \pi(h)$ for all $g, h \in G$. For $k \in \mathbb{N}$, let S_k denote the permutation group of the set $\{1, 2, \ldots, k\}$, as usual. Recall the action a of S_k on $\otimes^k V$, by permutation of the factors in k-fold tensor products, that was defined in class. Note that $a(\sigma)$, for $\sigma \in S_k$, is a linear transformation – in other words, a defines a representation of S_k on $\otimes^k V$. Show:

a) Any representation π of a group G on a vector space V induces a representation $\otimes^k \pi$ on $\otimes^k V$.

b) The representations $\otimes^k \pi$ of G and a of S_k commute – i.e., $\otimes^k \pi(g) \circ a(\sigma) = a(\sigma) \circ \otimes^k(g)$ for all $\sigma \in S_k$ and $g \in G$.