Math 55a, Fall 2004

First Assignment, Solutions
Adapted from Andrew Cotton

Problem 1. We define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as

$$f(x, y) = \frac{(x + y)(x + y + 1)}{2} + x$$

Claim: f is injective

Define $T_i = \frac{i(i+1)}{2}$. Now $0 \leq x \leq x + y$ implies $T_{x+y} \leq f(x, y) \leq T_{x+y+1} - 1$, which implies that $f(a, b) = f(c, d) \implies a + b = c + d$. Thus $a = c$, and so $b = d$ and $(a, b) = (c, d)$ and we're done.

Claim: f is surjective

Given a natural number n, there exists an $i \geq 0$ such that $T_i \leq n \leq T_{i+1} - 1$. Then $f(n - T_i, i + T_i - n) = n$. Note that, by construction, $n - T_i \geq 0$ and $i + T_i - n \geq 0$, and we're done.

Problem 2. Claim: Any countable set X has cardinality n for a unique $n \in \mathbb{N}$ or has cardinality $card(\mathbb{N})$.

Since X is countable, there exists an injection from X to $\mathbb{N}$, and so we can identify X with a subset of $\mathbb{N}$. If this subset has an upper bound of M, then X is finite and has cardinality less than or equal to M. If we define $f(x)$ for $x \in X$ to be the number of elements of X less than x (with X identified with a subset of the natural numbers), then we get a bijection with n for some $n \in \mathbb{N}$. This n is unique for there can be no bijective map between $\{1, \dots, n\}$ and $\{1, \dots, m\}$ by the pigeonhole principle. If the subset does not have an upper bound, then there exists a bijection between X and $\mathbb{N}$ by the same function $f(x)$, the number of elements of X less than x. This establishes the claim.

Now it suffices to consider $\mathbb{N}$ and sets of the form $\{1, \dots, n\}$ for $n \in \mathbb{N}$. $\mathbb{N}$ has a proper subset equal in cardinality to itself (the evens, for example, with the bijective map $f : \mathbb{N} \to 2\mathbb{N}$, $f(x) = 2x$). The set $\{1, \dots, n\}$ has no proper subset equal in cardinality to itself, again by the pigeonhole principle. This proves (a) and (b).

Problem 3. Let $M_n = Mor(\mathbb{N}, \{1, \ldots, n\})$. For $f \in M_n$ we define $g : M_n \to [0, 1]$ as

$$g(f) = \sum_{i=1}^{\infty} \frac{f(i) - 1}{n^{i+1}}$$

This function maps M_n surjectively onto $[0, 1]$ (for a given real number $x \in [0, 1]$, define $f(i)$ as the $(i+1)^t h$ term in the base n expansion of x). It fails to be injective only for decimals which terminate (call this set Y), as there will be another decimal expansion ending in $(n-1)s$ which also maps to this value (except when $x = 0$. It is easy to see that this is the only case in which it fails to be injective (if $f_1(i) \neq f_2(i)$ for some i and neither f_1 nor f_2 is constantly 1 afterwards, then $|g(f_1) - g(f_2)| \geq \frac{1}{n^k}$ where k is the first index after i before which both f_1 and f_2 have a value which is not 1).

This set (on which g fails to be injective) is countable, since it is a countable union of finite sets (the sets which contain all functions which are 1 after the $i^t h$ index.

Claim: $card(A \cup B) = card(A)$ for A infinite, B countable.
Since A is infinite, it has a subset C equal in cardinality $\aleph_0$ (create this by continually choosing elements, for example, and then index it with natural numbers). Create a bijective map from $A \cup B$ to A by mapping $A - C$ to itself and mapping $B \cup C$ to C (if B is infinite, map B to even indexed elements of C and map C to odd indexed elements of C, and if B is finite of cardinality n, map B to the first n elements of C and map C to the rest of C).

By the claim, $card([0, 1] \cup Y = card([0, 1))$. Now map $[0, 1)$ to $\mathbb{R}_{\geq 0}$ by $tan(\frac{\pi}{2}x)$. Similarly, $card([0, 1] \cup Y) = card((0, 1))$ which we then map to $\mathbb{R}$ by $tan(\pi x - \frac{\pi}{2})$. Note that $M_2 = 2^{\mathbb{N}}$, and so $card(2^{\mathbb{N}}) = card(\mathbb{R})$.