**Math 55a, Fall 2004**

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Fifth Assignment, Solutions
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**Note:** These solutions often use this simple fact proved in class: Suppose that $X$ is a topological space and that $S \subset X$, $x \in X$. Then $x \notin \text{clos}(S) \iff \exists$ open neighborhood $U_x$ of $x$ such that $S \cap U_x = \emptyset$. (The restriction "open" may also be removed from this condition.) Also, any open set $U$ containing a point $x$ is an open neighborhood of that point, because $x \in U = \text{int}(U)$.

**Problem 1.**

• *Every compact* $\left\{ \begin{array}{c} \textit{T-2} \\ \textit{T-3} \end{array} \right\}$ *space* $X$ *is* $\left\{ \begin{array}{c} \textit{T-3} \\ \textit{T-4} \end{array} \right\}$. *(Thus, every compact Hausdorff space is normal.)*

Suppose we have disjoint sets $S$, $T \subset X$, where $S$ is closed and $\left\{ \begin{array}{c} T \text{ contains one point } t \in X \\ T \text{ is closed} \end{array} \right\}$. We must prove there are disjoint sets $U \supset S$ and $V \supset T$. Because $X$ is $\left\{ \begin{array}{c} \text{T-2} \\ \text{T-3} \end{array} \right\}$, for each $s \in S$ there exist open disjoint sets $U_s \ni s$ and $V_s \supset T$. The $\{U_s\}$ cover $S$, which is a closed subset of a compact space and thus compact itself. Thus finitely many of the $\{U_s\}$ — say, $U_{s_1}$, $U_{s_2}$, ..., $U_{s_n}$ — cover $S$. Then consider the sets

$$U = \bigcup_{i=1}^{n} U_{s_i} \quad \text{and} \quad V = \bigcap_{i=1}^{n} V_{s_i}.$$

$U$ is the union of open sets and $V$ is the finite intersection of open sets, so they are both open themselves. Next, $S \subset U$ (since $U$ covers $S$) and $T \subset V$ (since $T$ is in each $V_{s_i}$). And finally, $U$ and $V$ are disjoint — each $u \in U$ lies in some $U_{s_i}$, and thus not in $V_{s_i}$ and not in $V$. This completes the proof.

• *Every metric space* $(X, d)$ *is normal.*

**First Solution:**

Given two disjoint, non-empty, closed subsets $S, T \subset X$, for each point $s \in S$ there exists an open set $U_s$ containing $s$ and disjoint from $T$. Then in this open set, there must be an epsilon-ball $B(s, \epsilon_s) \subset U_s$ also disjoint from $T$. Thus for any $t \in T$, we must have $\epsilon_s \le d(s,t)$. Now, write $B_s = B\left(s, \frac{\epsilon_s}{2}\right)$ and define

$$U_S = \bigcup_{s \in S} B_s.$$

As a union of open balls, $U_S$ is open. Define the values $\epsilon_t$ and the open set $U_T$ similarly. We claim that $U_S \cap U_T = \emptyset$. Suppose not. Then $\exists x$

such that $d(x,s) < \frac{\epsilon_s}{2}$ and $d(x,t) < \frac{\epsilon_t}{2}$ for some $s \in S, t \in T$. But then $d(s,t) \le d(x,t) + d(x,s) < \min(\epsilon_s, \epsilon_t) \le d(s,t)$, a contradiction. Thus $U_S$ and $U_T$ are indeed disjoint open sets containing $S$ and $T$, respectively.

**Second Solution:**
Let $S$ and $T$ be two non-empty, disjoint, closed subsets of $X$. For $x \in X$ define $d(x,S) = \inf_{s\in S} d(x,s)$. If $d(x,S) = 0$, then we can find $x_1, x_2, \ldots$ in $S$ such that $x_k \in d(x, \frac{1}{k})$ for $k \in \mathbb{N}$. Because $S$ is closed, the limit $x$ of $\{x_k\}$ must be in $S$ as well. Next, suppose that $\epsilon > 0$ is fixed. Then if $d(x,x') < \epsilon$, for each $s \in S$ we have $|d(x,s) - d(x',s)| < \epsilon$. Thus

$$d(x',S) \ge \inf_{s\in S}(d(x,s) - \epsilon) = d(x,S) - \epsilon$$

and similarly $d(x,S) \ge d(x',S) - \epsilon$. Therefore by the $\delta$-$\epsilon$ definition of continuity, $x \mapsto d(x,S)$ is continuous.

Define $d(x,T)$ analogously; as above, $x \mapsto d(x,T)$ is continuous, and if $d(x,T) = 0$ then $x \in T$.

Now consider the function $f : X \to \mathbb{R}$ such that

$$f(x) = \frac{d(x,S)}{d(x,S) + d(x,T)},$$

This function is defined, because we can never have $d(x,S) = d(x,T) = 0$ since $S \cap T = \emptyset$. Note that this function assumes the value 0 on $S$ and 1 on $T$ and is continuous (because the functions $d(x,S)$ and $d(x,T)$ are continuous, and because addition and division on the reals are continuous). Then by the converse of Urysohn's lemma (which is easier and done in class right before the lemma) we get that $X$ must be normal.

**Problem 2.**

We prove the statements slightly out of the order, in order to use (c) in the proof of (b).

(a) Suppose that a convergent net $(S, D, \ge)$ converged to distinct points $x$ and $y$. Because $X$ is Hausdorff, there exists disjoint open neighborhoods $N_x$ of $x$ and $N_y$ of $y$. Because the net converges to $x$, there exists $n_0 \in D$ such that $S(n) \in N_x$ for all $n \ge n_0$. Similarly, there exists $n_1 \in D$ such that $S(n) \in N_y$ for all $n \ge n_1$. Letting $n$ equal the upper bound of the set $\{n_0, n_1\}$, we find that $S(n) \in N_x \cap N_y = \emptyset$, a contradiction.

(c) Suppose by way of contradiction that some net $(S, D, \ge)$ in $B$ converges to a point $x \notin \mathrm{clos}(B)$. Because $x \notin \mathrm{clos}(B)$, there exists an open neighborhood $N_x$ of $x$ disjoint from $B$. Then by the definition of convergence, there is some $n' \in D$ such that $S(n') \in N_x$; but since $(S, D, \ge)$ is a net in $B$, we have $S(n') \in B$ as well. Thus $N_x$ and $B$ are not disjoint, a contradiction. Therefore our original assumption was false, and no convergent net in $B$ converges to a point outside $\mathrm{clos}(B)$.

Now suppose that $x \in \operatorname{clos}(B)$ — that is, every neighborhood of $x$ contains some element of $B$. Let $D$ be the set of these neighborhoods (whether open or not), where given sets $A_1, A_2 \in D$ we have $A_1 \geq A_2 \iff A_1 \subset A_2$. Also, for each $A \in D$ there is some point $b_A \in A \cap B$; using the axiom of choice, for each $A$ we may define $S(A) = b_A$ to be one such point. Then $(S, D, \geq)$ is a net because $(D, \geq)$ is clearly an order relation, and because given finitely many subsets $\{A_i\}$ in $D$, the set $\bigcap A_i$ is an upper bound of $\{A_i\}$. Then given any neighborhood $N_x$ of $x$, for all $N \geq N_x$ we have $S(N) \in N \subset N_x$ so that $(S, D, \geq)$ converges to $x$. Thus every point in $\operatorname{clos}(B)$ is the limit of some net in $B$. This completes the proof.

(b) If $B$ is closed, then from (c) no net in $B$ converges to a point not in $\operatorname{clos}(B) = B$. If $B$ is not closed then there is a point $x \in \operatorname{clos}(B) \setminus B \subset X \setminus B$; so from (c), some net in $B$ converges to $x$.

(d) **Note:** We assume that $D$ is nonempty, which is probably what Prof. Schmid wanted anyways.

Suppose that some subnet $(T, E, \geq_E) = (S \circ F, E, \geq_E)$ of $(S, D, \geq_D)$ converges to $x$; we prove that $x$ is a cluster point of $(S, D, \geq_D)$. Suppose we have $d \in D$ and a neighborhood $N_x$ of $x$. Since $(T, E, \geq_E)$ is a subnet, we know that there exists $k_0 \in E$ such that $k \geq_E k_0 \Rightarrow F(k) \geq_D d$. And since $(T, E, \geq_E)$ converges to $x$, we know there exists $k_1 \in E$ such that $k \geq_E k_1 \Rightarrow T(k) = S(F(k)) \in N_x$. But by the definition of a net, $\{k_0, k_1\}$ has some upper bound $k$; then letting $n = F(k)$, we have both $n \geq d$ and $S(n) \in N_x$. Therefore $x$ is indeed a cluster point of $(S, D, \geq_D)$, as desired.

Now suppose that $x$ is a cluster point of a net $(S, D, \geq_D)$. Let

$$E = \{(U, d) \mid U \text{ a neighborhood of } x, d \in D \text{ s.t. } S(d) \in U\}.$$

Also let $F : E \to D$ be defined by $F(U, d) = d$, and write $(U_1, d_1) \geq_E (U_2, d_2)$ iff $U_1 \subset U_2$ and $d_1 \geq_D d_2$.

It's easily shown that $\geq_E$ is reflexive, transitive, and antisymmetric (because both $\subset$ and $\geq_D$ are). Furthermore, given any finite subset $\{(U_1, d_1), \ldots, (U_n, d_n)\} \subset E$, let $U = \bigcap_{i=1}^n U_i$ and let $m \in D$ be an upper bound of $\{d_1, \ldots, d_n\}$ with respect to $\geq_D$. Then since $x$ is a cluster point, $\exists d \geq_D m \geq_D d_1, d_2, \ldots, d_n$ such that $S(d) \subset U$, so that $(U, d) \in E$ is an upper bound of $\{(U_1, d_1), \ldots, (U_n, d_n)\}$ with respect to $\geq_E$.

Thus, $(E, \geq_E)$ is a directed system and $(T, E, \geq_E) = (S \circ F, E, \geq_E)$ is a net in $X$. We already know $T = S \circ F$; and given $n_0 \in D$, then letting $k_0 = (X, n_0)$ we have $(U, d) \geq_E k_0 \Rightarrow F(U, d) = d \geq_D n_0$ by the definitions of $F$ and $\geq_E$. Thus $(T, E, \geq_E)$ is actually a *subnet* of $(S, D, \geq_D)$. And given any neighborhood $U$ of $x$ in $X$, because $x$ is a cluster point $\exists d \in D$ such that $S(d) \in U$. Writing $n_0 = (U, d)$, we know that $n = (U', d') \geq_E n_0 \Rightarrow T(n) \in U' \subset U$, so $(T, E, \geq_E)$ converges to $x$. Therefore if $x$ is a cluster point of a net, some subnet converges to $x$. This completes the proof.