Math 55a, Fall 2004

Sixth Assignment, Solutions
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Notation: Given a directed system $(D, \geq)$, consider the family $\mathcal{F}$ of finite nonempty subsets of D. By the axiom of choice, there exists a function $\overline{\text{up}}_{(D,\geq)} : \mathcal{F} \to D$ such that for all $E \in \mathcal{F}$, $\overline{\text{up}}_{(D,\geq)}(E)$ is an upper bound of the elements in E.

Problem 1.

Equivalently, we can prove that $(X, \mathcal{T})$ is compact if and only if every net in X has a cluster point. (From the last homework assignment, a net in X has a cluster point if and only if it has a convergent subnet.)

First suppose by way of contradiction that $(X, \mathcal{T})$ is compact, but that there is a net $(S, D, \geq)$ without a cluster point. Then for each $x \in X$, there exists an open set U_x containing x and an element $d_x \in D$ such that $\forall d \geq d_x, S(d) \notin U_x$. The sets $\{U_x \mid x \in X\}$ cover all of X, so there is some finite subcover $\{U_{x_k} \mid 1 \leq k \leq n\}$. But then for $d = \overline{\text{up}}_{(D,\geq)}(\{d_{x_1}, d_{x_2}, \ldots, d_{x_n}\})$, we have $d \geq d_{x_1}, d_{x_2}, \ldots, d_{x_n}$ so that $S(d) \notin U_{x_1}, U_{x_2}, \ldots, U_{x_n}$. Thus $S(d) \notin X$, a contradiction! So our original assumption must have been false, and if $(X, \mathcal{T})$ is compact then every net in X indeed has a cluster point.

Now suppose that $(X, \mathcal{T})$ is not compact, and let $\{U_\alpha \mid \alpha \in A\}$ be a cover of X with no finite subcover. Let D equal the family of all nonempty finite subsets of A. Since $\{U_\alpha\}$ has no finite subcover, for each set $d \in D$ there is some element in $X \setminus \bigcup_{\alpha \in d} U_\alpha$; let $S(d)$ equal one such element. Furthermore, write $d_1 \geq d_2$ if and only if $d_1 \supset d_2$. The relation $\geq$ is clearly reflexive, transitive, and antisymmetric. Also, given $d_1, d_2, \ldots, d_n \in D$, the set $d = \bigcup_{k=1}^n d_k$ is in D because it is also a nonempty finite subset of A; since $d_k \subset d$ for each $k = 1, 2, \ldots, n$, d is an upper bound of the d_k. Thus, $(D, \geq)$ is a directed system and $(S, D, \geq)$ is a net in X.

Now any $x \in X$ is in some U_α. For $d \geq \{\alpha\}$, we have $S(d) \in X \setminus \bigcup_{\alpha \in d} U_\alpha \subset X \setminus U_\alpha$ so that $S(d) \neq x$. Thus no x can be a cluster point. Therefore, if $(X, \mathcal{T})$ is not compact then some net in X has no cluster point. This completes the proof.

Note: Another possible solution uses the FIP characterization of compactness.

Problem 2.

Note: As proven in a previous assignment, to prove a function $F : X \to Y$ is continuous it suffices to prove it is continuous at each point — that is, for each $x \in X$ and each neighborhood $N_{F(x)} \subset Y$ of $F(x)$, there exists a neighborhood $N_x \subset X$ of X such that $F(N_x) \subset N_{F(x)}$. It is easily proven that it suffices to prove this last condition only for open sets $N_{F(x)}$, or even only for sets $N_{F(x)}$ in some neighborhood base of $N_{F(x)}$. We can similarly relax the burden of proof when showing that a net converges to a point: a net $(S, D, \geq)$ converges to x if every neighborhood of x contains $\{S(n) \mid n \geq n_0\}$ for some $n_0 \in D$, but it suffices to show that this condition holds for all neighborhoods in some neighborhood base of x.

For simplicity, we assume $C(X)$ is the set of continuous *complex*-valued functions; the proofs for real-valued functions are virtually identical.

(a) We first show that given a function h in a set $V_{f,K,\epsilon}$ we can find a "V-ball centered at h" in U (that is, some $V_{h,K',\delta} \subset U$).

Lemma 1. *If $h \in V_{f,K,\epsilon}$, then $\exists \delta > 0$ s. t. $V_{h,K,\delta} \subset V_{f,K,\epsilon}$.*

Proof: If $h \in V_{f,K,\epsilon}$, then $\alpha =_{\text{def}} \sup_{x\in K} |h(x) - f(x)| < \epsilon$. Define $\delta = \frac{\epsilon-\alpha}{2} > 0$, and consider the set $V_{h,K,\delta}$. If $g \in V_{h,K,\delta}$ then $\forall x \in K$,

$$\begin{aligned}
\sup_{x\in K} |g(x) - f(x)| &\leq \sup_{x\in K} (|g(x) - h(x)| + |h(x) - f(x)|) \\
&\leq \sup_{x\in K} |g(x) - h(x)| + \sup_{x\in K} |h(x) - f(x)| \\
&< \delta + \alpha = \epsilon,
\end{aligned}$$

so $g \in V_{f,K,\epsilon}$ as well. Therefore $V_{h,K,\delta} \subset V_{f,K,\epsilon}$, as desired. ■

Lemma 2. *A finite union of compact sets is compact.*

Proof: Suppose we have compact sets $C_1, \ldots, C_n$ and a cover $\{U_\alpha\}$ of these sets. Then $\{U_\alpha\}$ also covers each of $C_1, \ldots, C_n$, so for each C_i there is a finite subcover; combining these n finite subcovers gives a finite subcover for $C_1 \cup \cdots \cup C_n$, as desired. ■

Let $\mathcal{V}$ equal the family of sets of the form $V_{f,K,\epsilon}$. To verify that it is a base, we must check that every finite intersection of sets in $\mathcal{V}$ can also be written as a *union* of sets in $\mathcal{V}$.

The finite intersection of "no sets" is all of $C(X)$. Because X is locally compact, it contains some compact subset K_0. Then the union of the sets in $\mathcal{V}$ is all of $C(X)$ because every element $f \in C(X)$ is in $V_{f,K_0,1} \in \mathcal{V}$.

Now look at a finite intersection of one or more sets. Suppose we have

$$h \in V = V_{f_1,K_1,\epsilon_1} \cap \cdots \cap V_{f_n,K_n,\epsilon_n}.$$

Then for each i, from the first lemma $\exists \delta_i > 0$ such that $V_{h,K_i,\delta_i} \subset V_{f_i,K_i,\epsilon_i}$. Write $\delta = \min(\delta_1, \ldots, \delta_n) > 0$ and $K = \bigcup_{i=1}^n K_i$; from the second lemma,

K is compact. So $h \in V_{h,K,\delta}$, which itself is contained in V:

$$V_{h,K,\delta} \subset V_{h,K_i,\delta} \subset V_{h,K_i,\delta_i} \subset V_{f_i,K_i,\epsilon_i} \quad \text{for } i = 1, 2, \ldots, n.$$

Therefore V—the union of these sets $V_{h,K,\delta}$— is the union of sets in $\mathcal{V}$, as desired.

Now for fixed $h \in C(X)$, we verify that the $V_{h,K,\delta}$ ($K \subset X$ compact, $\delta > 0$) form a neighborhood base of f. Suppose that we have an open set $U \subset C(X)$ containing f. Then U is the union of sets of the form $V_{f,K,\epsilon}$, implying that h is one such set. It then follows from the first lemma that $h \in V_{h,K,\delta} \subset V_{h,K,\epsilon} \subset U$ for some K and δ, as desired. Setting $h = 0$, we find that the sets $U_{K,\epsilon}$ indeed form a neighborhood base of 0.

(b) We can define a Cauchy net in $(\mathbb{C}, ||)$ similarly to a Cauchy net in $C(X)$ — it is a net $(S, D, \geq)$ such that for every neighborhood $U \subset \mathbb{C}$ of 0, there exists $n_0 \in D$ such that $S(n) - S(m) \in U$ whenever $n, m \geq n_0$. (Equivalently, for all real $\epsilon > 0$, there exists $n_0 \in D$ such that $|S(n) - S(m)| < \epsilon$ whenever $n, m \geq n_0$.)

Lemma 3. *Any Cauchy net $(S, D, \geq)$ in $\mathbb{C}$ converges to a unique complex number.*

Proof: For each $i = 1, 2, \ldots, \exists n_i$ s. t. $m, n \geq n_i \Rightarrow |S(n) - S(m)| < \frac{1}{i}$. Define $t_i = \overline{\text{up}}_{(D,\geq)}(\{t_1, t_2, \ldots, t_{i-1}, n_i\})$ for $i \in \mathbb{N}$. Then $t_1 \leq t_2 \leq t_3 \leq \ldots$ so that $j \geq k \Rightarrow |S(t_j) - S(t_k)| < \frac{1}{k}$. Thus $\{S(t_i)\}$ is a Cauchy sequence in $\mathbb{C}$ and therefore converges to some value $r \in \mathbb{C}$.

Then for each ϵ, there exists n_j such that $m, n \geq n_j \Rightarrow |S(m) - S(n)| < \frac{\epsilon}{2}$. At the same time, since $\{S(t_i)\}$ converges to r there exists $t_k \geq t_j \geq n_j$ such that $|S(t_k) - r| < \frac{\epsilon}{2}$. Then for all $n \geq t_k$ we have

$$|S(n) - r| \leq |S(n) - S(t_k)| + |S(t_k) - r| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so that $(S, D, \geq)$ indeed converges to r.

Furthermore, because $\mathbb{C}$ is Hausdorff, $(S, D, \geq)$ converges to a unique number. ■

Now given any $\epsilon > 0$ and any compact set $K \in X$, define $n_{K,\epsilon} \in D$ such that $S(m) - S(n) \in U_{K,\epsilon}$ when $m, n \geq n_{K,\epsilon}$. Now given a fixed $x_0 \in X$, look at $(T, D, \geq)$ where $T(d) = S(d)(x_0)$. Since X is locally compact, there is a compact neighborhood K_0 of x_0. Then for every $\epsilon > 0$, when $m, n \geq n_{K,\epsilon}$ we have

$$\begin{aligned} |T(m) - T(n)| &= |S(m)(x_0) - S(n)(x_0)| \\ &\leq \sup_{k \in K_0} |S(m)(x) - S(n)(x)| \\ &< \epsilon. \end{aligned}$$

Thus $(T, D, \geq)$ is a Cauchy net in $\mathbb{C}$ and hence converges to some unique value $y_0 \in \mathbb{C}$. Define $f : X \to \mathbb{C}$ by $f(x_0) = y_0$ for each x_0 and corresponding y_0. A standard argument using the triangle inequality shows that $|S(n)(x_0) - f(x_0)| \leq \epsilon$ for all $n \geq n_{K_0,\epsilon}$.

By the definition of f, $(S, D, \geq)$ "converges to f pointwise." We must prove that $f \in C(X)$ and that $(S, D, \geq)$ "converges to f as a whole."

- *$f \in C(X)$.*

Let $B(z, \epsilon)$ denote the ϵ-ball in $\mathbb{C}$ centered at z. To prove $f \in C(X)$, it suffices to prove that given any $x_0 \in X$ and $B(f(x_0), \epsilon) \subset \mathbb{C}$, there is an open $\mathcal{O}$ containing x_0 such that $f(\mathcal{O}) \subset B(f(x_0), \epsilon_0)$.

First, since X is locally compact, we can find a compact neighborhood $K_0 \subset X$ containing x_0. Also, write $\epsilon = \epsilon_0/3$.

From above, we know that $|f(x) - S(n_{K,\epsilon})(x)| \leq \epsilon$ for all $x \in K$. Also, since each $S(n)$ is continuous we know there is an open neighborhood T of x_0 such that $y \in T \Rightarrow |S(n_{K,\epsilon})(y) - S(n_{K,\epsilon})(x_0)| < \epsilon$. Then the set $U = T \cap \mathrm{int}(K_0)$ is open (since it is the intersection of two open sets) and contains x_0. So for $y \in U$ and $n \geq n_{K,\epsilon}$, we have

$$\begin{aligned}
|f(x_0) - f(y)| &\leq |f(x_0) - S(n_{K,\epsilon})(x_0)| \\
&\quad + |S(n_{K,\epsilon})(x_0) - S(n_{K,\epsilon})(y)| \\
&\quad + |S(n_{K,\epsilon})(y) - f(y)| \\
&< \epsilon + \epsilon + \epsilon \\
&= \epsilon_0.
\end{aligned}$$

Thus there is an open neighborhood $\mathcal{O}$ containing x_0 such that $f(\mathcal{O}) \subset B$, as desired. This completes the proof!

- *$(S, D, \geq)$ converges to f.*

Given a set $V_{f,K,\epsilon}$, choose a positive $\epsilon' < \epsilon$ and set $n_0 = n_{K,\epsilon'}$ suffices. Then $|S(n)(x) - f(x)| \leq \epsilon'$ for all $n \geq n_0, x \in K$. Thus, for all $n \geq n_0$,

$$\sup_{x \in K} |S(n)(x) - f(x)| \leq \epsilon' < \epsilon,$$

implying that $\{S(n) \mid n \geq n_0\} \subset V_{f,K,\epsilon}$. Because the sets $V_{f,K,\epsilon}$ constitute a neighborhood base of f, this proves that $(S, D, \geq)$ converges to f.

Note: Here is a sketch of a slightly different solution. In class it was proved that $C(X)$ is complete when X is compact. More generally, every Cauchy net $(S, D, \geq)$ in $C(X)$ converges to a unique function in $C(X)$, and it converges to that same function pointwise — we can construct this limit by constructing a convergent sequence as in the proof about "real Cauchy nets" above, and then prove that $(S, D, \geq)$ converges to the limit as in the last portion above. Now if X is only locally compact, given a net $(S, D, \geq)$ in $C(X)$ we can consider two variants of this net. First, for any $x \in X$ we can consider the net $(S_x, D, \geq)$ in $\mathbb{C}$ where $S_x(d) = S(d)(x)$. Second, for any compact $K \subset X$ we can consider the net $(S_K, D, \geq)$ where $S_K(d) = S(d)|_K$. Given K, this latter net is a Cauchy net in $C(K)$ (why?) and hence has a unique limit $f_K \in C(K)$ that the net converges to both as a whole and pointwise; that is, $(S_x, D, \geq)$ converges to to the unique limit $f_K(x)$ for each $x \in K$. Because each $x \in X$ is contained in some compact $K \subset X$, and because $f_K(x)$ is the same over all compact K containing x, there is a well-defined function $f : X \to \mathbb{C}$ by $f(x) = f_K(x)$ for some such K. To prove this function is actually in $C(X)$, given $x \in X$ and a neighborhood of $N_{f(x)}$, let K be any compact neighborhood of x; then there exists $\mathcal{O}_1$ containing x such that $f(\mathcal{O}_1) \subset N_{f(x)}$, where N_x is an open neighborhood of x *viewed as a subset of* K. But this means that we can write $\mathcal{O}_1 = K \cap \mathcal{O}_2$ for open $\mathcal{O}_2 \subset X$, and then the open neighborhood $N_x = \int(K) \cap \mathcal{O}_2 \subset X$ containing x satisfies $f(N_x) \subset N_{f(x}$, as needed.

Note that with this argument, we do not have to prove that $f \in C(X)$ for X locally compact using the same ϵ arguments that were used in class for X compact. Rather, this proof proves a claim for compact spaces, then uses that result to prove the claim for locally compact spaces by creating "overlapping functions." Similar overlapping functions will appear later when we study calculus on manifolds.

(c) Because addition and multiplication are continuous on the complex numbers, given $M > 0$ they are uniformly continuous on the compact set $\{a \mid |a| \leq M\} \times \{b \mid |b| \leq M\} \subset \mathbb{C} \times \mathbb{C}$.

We prove a similar result for reciprocation: given $M, \epsilon > 0$, $\exists \delta > 0$ such that for any $a \in \mathbb{C}$ with $|a| \geq M$, we have $\frac{1}{B(a,\delta)} \subset B(\frac{1}{a}, \epsilon)$. Choose $\delta > 0$ so that $M - \delta > 0$ and $\frac{\delta}{M(M-\delta)} < \epsilon$. This suffices because then

$$a_1 \in B(a, \delta) \Longrightarrow \left| \frac{1}{a_1} - \frac{1}{a} \right| = \frac{|a - a_1|}{|aa_1|} < \frac{\delta}{M(M-\delta)} < \epsilon \Longrightarrow \frac{1}{a_1} \in B\left(\frac{1}{a}, \epsilon\right).$$

We now prove that multiplication from $C(X) \times C(X)$ to $C(X)$ is continuous; the proof for addition is virtually identical. It suffices to produce, given arbitrary $f, g \in C(X)$ and $W = V_{f \cdot g, K, \epsilon} \subset C(X)$, an open neighborhood of (f, g) whose image under multiplication is in W. Because f and g are continuous functions and K is compact in X, $f(K)$ and $g(K)$ are compact in $\mathbb{C}$ and thus are bounded. Hence, $\exists M \in \mathbb{R}$ such that $|f(x)|, |g(x)| \leq M$ for all $x \in K$. Then from our initial observations, there exists $\delta > 0$ such that $B(f(x), \delta) \cdot B(g(x), \delta) \subset B(f(x)g(x), \frac{\epsilon}{2})$ for all $x \in K$.

Now consider the open set $V_1 \times V_2$, where $V_1 = V_{f,K,\frac{\delta}{2}}$ and $V_2 = V_{g,K,\frac{\delta}{2}}$. If $f_1 \in V_1$ and $g_1 \in V_2$, then for each $x \in K$ we have $f_1(x) \in B(f(x), \delta)$ and $g_1(x) \in B(g(x), \delta)$ so that $f_1(x) \cdot g_1(x) \in B(f(x) + g(x), \frac{\epsilon}{2})$. Hence, $V_1 \times V_2$ is an open neighborhood of (f_1, g_1) whose image under multiplication is in W. Therefore multiplication (and similarly, addition) is indeed continuous.

A fairly similar argument shows that reciprocation is continuous; suppose we have $f \in C^*(X)$ and $V_{\frac{1}{f},K,\epsilon} \cap C^*(X) \subset C^*(X)$. Since f and the absolute value (from $\mathbb{C}$ to $\mathbb{R}$) are continuous, and K is compact in X, $|f(K)|$ is bounded. Hence, $\exists M \in \mathbb{R}$ such that $|f(x)| \leq \frac{1}{M}$ and $|\frac{1}{f(x)}| \geq M$ for all $x \in K$. Then, from our initial observations, there exists $\delta > 0$ such that $\frac{1}{B(f(x),\delta)} \subset B\left(\frac{1}{f(x)}, \frac{\epsilon}{2}\right)$ for all $x \in K$.

Now consider the open set $V = V_{f,K,\delta} \cap C^*(X)$. If $f_1 \in V$, then for each $x \in K$ we have $f_1(x) \in B(f(x), \delta)$ so that $\frac{1}{f_1(x)} \in B\left(\frac{1}{f_1(x)}, \frac{\epsilon}{2}\right)$. Hence, $f_1 \in V$ and $f(V) \subset W$, as desired.

(d) Write V-balls in $C(X)$ as $V^{(X)}_{f,K,\epsilon}$ and V-balls in $C(Y)$ as $V^{(Y)}_{f,K,\epsilon}$. It suffices to produce, given any $f \in C(Y)$ and open $W = V_{F^*f,K,\epsilon} \subset C(X)$, an open set $U \in F^{*-1}(W)$ containing f. Because F is continuous, $F(K)$ is compact so that $U = V^{(Y)}_{f,F(K),\epsilon}$ is a valid V-ball in $C(Y)$. Suppose that $f_1 \in U$. Then for all $x \in K$, we have $F(x) \in F(K)$. Thus,

$$\begin{aligned} \sup_{x \in K} |F^*(f_1)(x) - F^*(f)(x)| &= \sup_{x \in K} |f_1 \circ F(x) - f \circ F(x)| \\ &\leq \sup_{y \in F(K)} |f_1(y) - f(y)| \\ &< \epsilon \end{aligned}$$

and $F^*(f_1) \subset W$. Therefore, $F^*(U) \subset V_1$, as needed.