Math 55a, Fall 2004

Second Assignment, due October 5

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1. The completeness of $\mathbb{R}$, considered as a metric space in the usual way, is equivalent to the the least upper bound axiom – equivalent modulo the various algebraic properties of $\mathbb{R}$ and properties of the order $\geq$. Prove this equivalence. You may substitute the "Dedekind cut" axiom for the least upper bound axiom, if you wish.

2. Let (X, d_X), (Y, d_Y) be metric spaces, and $F : X \to Y$ a map between them. Establish the equivalence of the following properties of F.
a) For every open set $U \subset Y$, the inverse image $F^{-1}(U)$ is open in X (this is the usual definition of continuity).
b) For every closed set $S \subset Y$, the inverse image $F^{-1}(S)$ is closed.
c) For every $x_0 \in X$ and every $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, x_0) < \delta$ implies $d_Y(F(x), F(x_0)) < \epsilon$ (this is the ϵ-δ definition of continuity).
d) For every convergent sequence $\{x_k\}$ in X, the image sequence $\{F(x_k)\}$ converges, and $\lim_{k\to\infty} F(x_k) = F(\lim_{k\to\infty} x_k)$ (this is the definition of continuity in terms of sequences).

3. Let $C \subset [0, 1]$ denote the Cantor set – you can find the definition in *Simmons*, and in many other textbooks. By construction, the complement $U = [0, 1] - C$ is a disjoint union of a countable number of open intervals I_k.
a) Compute the total length of the I_k.
b) Let $\chi_U : [0, 1] \to \mathbb{R}$ denote the characteristic function of U; i.e., $\chi_U(x) = 1$ if $x \in U$ and $\chi_U(x) = 0$ if $x \in C$. Prove that χ_U is Riemann integrable, *in the technical sense*, over the interval $[0, 1]$, and compute the integral.