Math 55a, Fall 2004

Third Assignment, due October 12

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1. This problem involves a large number of verifications. Try to be succinct: you need not include every last detail; on the other hand, there must be enough information to provide a clear idea of the arguments. Let p be a prime number, and let $\mathbb{Q}_p$, $\mathbb{Z}_p$ denote the sets of p-adic numbers, respectively the set of p-adic integers. In other words, $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value, and $\mathbb{Z}_p$ the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$. Further notation: S_p is the set $\{0, 1, \ldots, p-1\}$.
a) Show that the algebraic operations on $\mathbb{Q}$ – addition, multiplication, taking the negative and the reciprocal of a non-zero number – extend naturally to $\mathbb{Q}_p$. Also show that the p-adic absolute value extends.
b) Verify that these algebraic operations turn $\mathbb{Q}_p$ into a field – i.e., addition and multiplication are associative, commutative, have neutral elements as well as inverses, and the distributive law holds.
c) Show: every strictly positive integer n can be expressed uniquely as a finite sum $n = \sum_{0 \le k \le N} a_k \, p^k$, with coefficients $a_0, a_1, \ldots a_N$ in the set S_p.
d) Show: there exists a convergent series $s = \sum_{0 \le k < \infty} a_k \, p^k$, with $a_0, a_1, \cdots \in S_p$, such that $s = -1$ (Hint: $-1 = (p-1)(1-p)^{-1}$).
e) Show: if n is a positive integer relatively prime to p, there exists a convergent infinite series $s = \sum_{0 \le k < \infty} a_k \, p^k$, with $a_0, a_1, \cdots \in S_p$, such that $s = 1/n$.
f) Deduce that every $q \in \mathbb{Q}_p$ can be expressed uniquely as a finite or infinite series $q = \sum_{v \le k} a_k \, p^k$, with $a_v, a_{v+1}, \cdots \in S_p$ and $v = v_p(q) =$ p-adic valuation of q, and that conversely every such series represents a p-adic number. Prove that such series can be added and multiplied (careful: S_p is not closed under multiplication), and that this formal addition and multiplication coincides with the operations defined in a).
g) Prove that the ring of p-adic integers $\mathbb{Z}_p$ is both compact and open in $\mathbb{Q}_p$. Is $\mathbb{Q}_p$ compact?

2. Define a distance function d on $X =_{\mathrm{def}}$ set of sequences in $[0, 1]$ by the formula $d(\{x_k\}, \{y_k\}) = \sup_k |x_k - y_k|$, and show that d has all the required properties of a metric. Then show that the metric space (X, d) is bounded – i.e., there exists some constant $M > 0$ such that the distance between any two points in X is bounded by M – but not totally bounded. Is every totally bounded metric space bounded?