Math 55a, Fall 2004

Fourth Assignment, due October 19

1. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, $\mathcal{S}_Y$ a sub-base for the topology $\mathcal{T}_Y$, and $F : X \to Y$ a map between the two spaces. By definition, a *neighborhood* of a point in a topological space is a set – not necessarily an open set! – whose interior contains the point in question. By definition, F is continuous at a given point $x \in X$ if, for every neighborhood $N_{F(x)}$ of $F(x)$, there exists a neighborhood N_x of x such that $F(N_x) \subset N_{F(x)}$. Show that the following conditions on F are equivalent:

i) F is continuous (i.e., $F^{-1}(U) \in \mathcal{T}_X$ for every $U \in \mathcal{T}_Y$).
ii) $F^{-1}(S)$ is closed for every closed set $S \subset Y$.
iii) $F^{-1}(V) \in \mathcal{T}_X$ for every $V \in \mathcal{S}_Y$.
iv) F is continuous at every point $x \in X$.

2. In this problem, $\mathcal{T}_{\text{reg}}$ denotes the usual topology on the real line $\mathbb{R}$, and $\mathcal{T}_{\text{cof}}$ the cofinite topology on $\mathbb{R}$ – the topology for which a set, other than $\mathbb{R}$ itself, is closed if and only it is finite or empty.
a) Is $\mathcal{T}_{\text{cof}}$ a Hausdorff topology?
b) Characterize the sequences in $\mathbb{R}$ which are convergent with respect to the $\mathcal{T}_{\text{cof}}$ topology, in as simple terms as possible.
c) Characterize the functions $f : \mathbb{R} \to \mathbb{R}$ which are continuous with respect to the topology $\mathcal{T}_{\text{cof}}$ on the domain and the topology $\mathcal{T}_{\text{reg}}$ on the range, in as simple terms as possible.

3. Recall the properties of $\mathbb{Q}_p$ and $\mathbb{Z}_p$ which were established in the last assignment. Define the ring of adeles, $\mathbb{A}$, as follows. As a set, $\mathbb{A}$ is contained in the Cartesian product of $\mathbb{R}$ and the $\mathbb{Q}_p$, for all primes p:

$$\mathbb{A} = \{\, a \in \mathbb{R} \times \textstyle\prod_p \mathbb{Q}_p \mid\ a_p \in \mathbb{Z}_p \text{ for all but finitely many primes } p \,\}.$$

Equivalently, $\mathbb{A} = \cup_S \mathbb{A}_S$ is the union, extended over all finite sets S of prime numbers, of

$$\mathbb{A}_S \;=\; \{\, a \in \mathbb{R} \times \textstyle\prod_p \mathbb{Q}_p \mid\ a_p \in \mathbb{Z}_p \text{ for all } p \notin S \,\} \;\cong\; \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p \;.$$

Equip each $\mathbb{A}_S \cong \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p$ with the product topology, and decree that a subset $U \subset \mathbb{A}$ is open if and only if $U \cap \mathbb{A}_S$ is open for every S. Define the addition and multiplication of adeles component-by-component.
a) Show that $\mathbb{A}$ is a locally compact Hausdorff space (a topological space is locally compact if every point has a compact neighborhood).
b) Show that addition and multiplication, viewed as maps from $\mathbb{A} \times \mathbb{A}$ to $\mathbb{A}$, are continuous.